MEASURING RISK AVERSION IN NON-LINEAR PREFERENCE MODELS

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The paper introduces the so-called natural preference relation on sets of risks and studies a concept of risk aversion within this context. A representation theorem for natural preferences is provided, and a method of quantitative measurement of risk aversion is studied.

Keywords: risk, decision-making, natural preference, risk aversion, representation theorem, coherent risk measure

1. Introduction

Risk aversion has been studied in [1,2,3] within the framework of expected utility [4]. More general concept of risk aversion was introduced and studied in [5] for distorted probability model. The current paper is devoted to studying risk aversion in a non-linear preference model called natural. Section 2 recalls basic concepts. Section 3 is devoted to representation theorem for preference relation under consideration. Section 4 introduces risk aversion concept and methods of its calculation. In section 5 a few examples of risk aversion calculation are presented for the classic coherent risk measures.

2. Basic concepts

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, and $\mathcal{X}$ be a set of random variables defined on $(\Omega, \mathcal{A}, \mathbf{P})$, and possessing finite moments of orders up to some value p, $1 \le p \le \infty$. Elements of $X \in \mathcal{X}$, will be called *risks* in accordance with tradition [6,7]. In particular, any risk $X \in \mathcal{X}$ possesses finite expectation. We would not distinguish between random variables which coincide almost surely. To be precise, let us call random variables X, Y equivalent ($X \sim Y$) if $\mathbf{P}(X \neq Y) = 0$. Moreover we will think that standard transform to factor set has been already implemented, and the notation $\mathcal{X}$ for the factor set. $\mathcal{X}$ may be endowed with a norm

$$\|X\| = \left(\int_\Omega |X|^p(\omega) d\mathbf{P}(\omega) \right)^{1/p}$$

and a natural ordering

$$X \le Y \Longleftrightarrow \mathbf{P}\left(\omega : \ X(\omega) \le Y(\omega)\right) = 1.$$

Denote I the identical unity

$$I(\omega) = 1, \ \omega \in \Omega.$$

Risks aI with real values a possess degenerate distributions, so we will call them degenerate risks. Next, denote $\mathcal{X}_0$ the set of zero mean risks

$$\mathcal{X}_0 = \{X \in \mathcal{X} : \ \mathbf{E}X = 0\}.$$

Preference relation $\preceq$ on $\mathcal{X}$ is any complete transitive binary relation. Its symmetric part is *equivalence*

$$X \sim Y \Longleftrightarrow X \preceq Y, \ Y \preceq X,$$

and asymmetric part forms a strict preference relation

$$X \prec Y \Longleftrightarrow X \preceq Y, \ Y \npreceq X.$$

Upper class $U(X)$ of a risk X with respect to $\preceq$ is the set

$$U(X) = \{Y \in \mathcal{X} : \ X \preceq Y\}.$$

Lower class is defined similarly

$$L(X) = \{Y \in \mathcal{X} : \ Y \preceq X\}.$$

Intersection of these two classes defines the equivalence class for a risk X:

$$U(X) \cap L(X) = \{Y \in \mathcal{X} : \; Y \sim X\},$$

which is denoted by $K(X)$. Equivalence $\sim$, as usual, generates a partition of $\mathcal{X}$ to equivalence classes, the collection of which is called factor set $\mathcal{X}/\sim$. Here K may be regarded as a mapping from $\mathcal{X}$ to $\mathcal{X}/\sim$, which maps a risk X to its equivalence class $K(X)$. We will call the mapping K *partitioning*.

Preference relation $\preceq$ on $\mathcal{X}$ is called *finite*, if $\mathcal{X}$ lacks "infinitely good " and "infinitely bad " risks, that is, for any $Y \in \mathcal{X}$ upper and lower classes $U(Y), L(Y)$ contain degenerate risks. Preference relation is *monotone*, if

$$X \leq Y \Longrightarrow X \preceq Y.$$

Monotone preference is called *strictly monotone*, if

$$\mathbf{P}(X < Y) = 1 \Longrightarrow X \prec Y.$$

Note that for a strictly monotone preference $a < b$ implies $aI \prec bI$.

Monotone preference relation $\preceq$ is called *lower semi-continuous* at a point $Y \in \mathcal{X}$, if for any countable family of risks $A \subseteq U(Y)$ the inclusion $\inf_{X \in A} X \in U(Y)$ holds. Similarly, a monotone preference relation $\preceq$ is called *upper semi-continuous* at a point $Y \in \mathcal{X}$, if for any countable set of risks $A \in L(Y)$ the inclusion $\sup_{X \in A} X \in L(Y)$ holds. A preference relation is *continuous* on $\mathcal{X}$, if it is both lower and upper semi-continuous at each point $Y \in \mathcal{X}$. Preference relation $\preceq$ on $\mathcal{X}$ is called *non-saturated*, if for any $X, Y \in \mathcal{X}$ there exist constants $c < d$ such that $X + cI \prec Y \prec X + dI$.

Definition 1. A continuous finite strictly monotone non-saturated preference relation is called *natural*.

The set of admissible risks $A_{\preceq}$ for a given preference relation $\preceq$ is defined by

$$A_{\preceq} = \{X \in \mathcal{X} : \; 0 \preceq X\} = U(0).$$

For a monotone preference this set contains non-negative cone $C_+ = \{X \in \mathcal{X} : \; X \geq 0\} \subseteq A_{\preceq}$. If the preference under consideration is strictly monotone then the negative cone

$$C_{--} = \{X \in \mathcal{X} : \; \mathbf{P}(\omega : \; X(\omega) < 0) = 1\}$$

does not intersect with $A_{\preceq}$, that is, $C_{--} \cap A_{\preceq} = \emptyset$.

The upper class may be represented as follows.

Theorem 1. Let $\preceq$ be a natural preference relation on $\mathcal{X}$. Then

$$U(X) = \bigcup_{Y \in K(X)} (Y + C_+), \tag{1}$$

where sum should be understood in Minkowski sense.

Proof. Denote $\tilde{U}(X)$ the right hand side of (1). Let $Z \in \tilde{U}(X)$. Then for some $Y \in K(X)$, $V \in C_+$ we have $Z = Y + V$. Since $Y \leq Z$, monotonicity of $\preceq$ implies $X \sim Y \preceq Z$, thus $Z \in U(X)$ and $\tilde{U}(X) \subseteq U(X)$.

Now let $Z \in U(X)$. Since $\preceq$ is non-saturated, we have $Z - cI \preceq X$ for some $c \geq 0$. Next, continuity of $\preceq$ implies $Y = Z - dI \in K(X)$ and $Z = Y + dI$ for some $0 \leq d \leq c$, which means $Z \in Y + C_+$, thus $Y \in \tilde{U}(X)$. The proof is complete. □

We will call a preference relation *convex* if its upper classes $U(X)$ are convex for each $X \in \mathcal{X}$.

3. Preference representation

In [8] theorems on preference representation in a set of probability distribution are presented. Here we will state similar results for preferences in sets of random variables.

A real valued functional $f : \mathcal{X} \to R$ represents $\preceq$ if

$$X \preceq Y \Longleftrightarrow f(X) \leq f(Y). \tag{2}$$

Recall [8] that along with f the preference relation $\preceq$ is represented by any functional g which is linked to f by a strictly increasing transform h:

$$g(X) = h(f(X)),\ X \in \mathcal{X}.$$

So representing functional, if it exists, is surely *not the* representing functional.

Theorem 2. Let $\preceq$ be a natural preference relation on $\mathcal{X}$. Then there exists a real functional $f = f_{\preceq}$ representing the preference $\preceq$.

Proof. Let $\mathcal{X}/\sim$ be the factor set of $\mathcal{X}$ with respect to the symmetric part of $\preceq$. Since the preference is natural, each equivalence class contains at most one degenerate random variable of the form aI. We will show that one such risk is always present in each equivalence class.

Fix $Y \in \mathcal{X}$ and denote $A = \{a \in R:\ aI \in L(Y)\}$, $B = \{b \in R:\ bI \in U(Y)\}$. Since the preference is finite, the sets A, B are not empty. Moreover, monotonicity of $\preceq$ implies $a \leq b$ for any $a \in A, b \in B$. Now continuity of $\preceq$ implies $\sup A = \inf B$. Denote the common value of the extrema c. Consider any sequence $a_n \in A$ such that $\lim_{n\to\infty} a_n = c$. Upper semi-continuity gives $\sup_n a_n I = cI \in L(Y)$. Similarly consider a sequence $b_n \in B$ such that $\lim_{n\to\infty} b_n = c$, by lower semi-continuity we have $\inf_n b_n I = cI \in U(Y)$. So $cI \in L(Y) \cap U(Y) = K(Y)$, and the equivalence class $K(Y)$ contains the degenerate risk cI as required.

So each equivalence class from $\mathcal{X}/\sim$ contains the degenerate risk of the form cI, where $c = c_Z$ depends on the equivalence class $Z \in \mathcal{X}/\sim$. A representing functional may be written as

$$f(X) = c_{K(X)},\ X \in \mathcal{X},$$

where K is the partitioning mapping from $\mathcal{X}$ to $\mathcal{X}/\sim$. We will call such representing functional *certainty equivalent* of the preference $\preceq$. The proof is complete. □

Note that certainty equivalent f has a property

$$f(cI) = c,\ c \in R.$$

A natural preference relation $\preceq$ is called positively homogeneous, if its certainty equivalent is positively homogeneous:

$$f(\lambda X) = \lambda f(X),\ \lambda \geq 0,\ X \in \mathcal{X}.$$

4. Risk aversion

Preference relation describes personal attitude to risk. Most persons are characterized by risk aversion. In the present paper we give a definition and methods for quantitative measurement of the concept.

Definition 2. A person with preference relation $\preceq$ is risk averse if for any $x \in R$ and any $\Delta \in \mathcal{X}_0$, $\Delta \neq 0$ we have

$$xI + \Delta \prec xI. \tag{3}$$

This means that adding a pure risk Δ to a certain xI makes things worse (from the personal point of view).

Preference relation $\preceq$ is *shift invariant* if

$$X \preceq Y \Longrightarrow X + aI \preceq Y + aI,\ a \in R.$$

For an invariant preference the concept may be defined as $\Delta \prec 0$ for any $\Delta \in \mathcal{X}_0$, $\Delta \neq 0$.

Upper classes of degenerate risks under shift invariant preference have similar form and are shifts of each other:

$$U(cI) = cI + U(0),\ c \in R.$$

The same is true for lower classes

$$L(cI) = cI + L(0),\ c \in R.$$

One might say that risk aversion under shift invariant preference does not depend on initial capital.

I risk aversion grows with capital, then the upper class becomes narrower, that is, $a < b$ implies

$$U(bI) \subseteq (b-a)I + U(aI).$$

A person with such preference may be called “greedy”. When initial capital grows, the person becomes less risky. Opposite behavior is accompanied with enlarging of upper classes as initial capital grows:

$$U(bI) \supseteq (b-a)I + U(aI),\ a < b.$$

Let us endow $\mathcal{X}$ with a norm $\|\cdot\|$ and denote $B = \{X \in \mathcal{X} :\ \|X\| \le 1\}$ the unit ball of the space $\mathcal{X}$. The norm may be calibrated so that

$$\|I\| = 1. \tag{4}$$

Denote $B_0 = \mathcal{X}_0 \cap B$ and define the quantity for measuring risk aversion.

Definition 3. For a given natural preference $\preceq$ on $\mathcal{X}$ risk aversion is the function

$$r(x) = \sup_{\Delta \in B_0} \left(f(xI) - f(xI + \Delta)\right),\ x \in R, \tag{5}$$

where f stands for the certainty equivalent of $\preceq$.

The concept of risk aversion allows the following dual description.

Theorem 3. Let $\preceq$ be a natural preference with certainty equivalent f and upper classes U. Then the function

$$r_1(x) = x - \sup\{y :\ U(yI) \supseteq xI + B_0\},\ x \in R \tag{6}$$

coincides with $r(\cdot)$ in (5).

Proof. Note that (5) may be rewritten as

$$r(x) = x - \inf_{\Delta \in B_0} f(xI + \Delta),\ x \in R \tag{7}$$

and show that (7) is equivalent to (6). To do that it is sufficient to prove the equality

$$\sup\{y :\ U(yI) \supseteq xI + B_0\} = \inf_{\Delta \in B_0} f(xI + \Delta) \tag{8}$$

for each fixed $x \in R$. Denote $A(x) = \{y :\ U(yI) \supseteq xI + B_0\}$. For $y \in A(x)$, $\Delta \in B_0$ we have $yI \preceq xI + \Delta$, in other words, $y \le f(xI + \Delta)$, so

$$\sup A(x) \le \inf_{\Delta \in B_0} f(xI + \Delta). \tag{9}$$

On the other hand, by definition of $A(x)$, for any $y' \notin A(x)$ there is a $\Delta' \in B_0$ such that $y'I \succ xI + \Delta$, that is, $y' > f(xI + \Delta')$. Therefore inequality in (9) Is in fact equality. The proof is complete.. □

Risk aversion for shift invariant preference relation does not depend on x and has the form

$$r = \sup_{\Delta \in B_0} |f(\Delta)|. \tag{10}$$

Theorem 4. Let $\preceq_1$, $\preceq_2$ be two natural preference relations on $\mathcal{X}$, and U_1, U_2 and r_1, r_2 be the corresponding upper classes and risk aversion functions. Suppose that upper classes satisfy $U_1(xI) \subseteq U_2(xI),\ x \in R$. Then risk aversion functions satisfy the inequality

$$r_1(x) \ge r_2(x),\ x \in R. \tag{11}$$

Proof. The theorem is a direct corollary of representation established in theorem 3. □

Note that for a natural preference relation the minimum upper class for any risk $X \in \mathcal{X}$ is the cone $X + C_+$ (such a preference relation is shift invariant). Theorem 4 states that such a preference corresponds to maximum risk aversion.

5. Coherent risk measures

To provide an example let us calculate risk aversion for preferences generated by the so called coherent risk measures [6,10]. Recall that a functional $f : \mathcal{X} \to R$ is called a *coherent risk measure* if it is monotone $X \le Y \Longrightarrow f(X) \le f(Y)$, positive homogeneous $f(\lambda X) = \lambda f(X),\ \lambda \ge 0$, shift invariant $f(X+aI)=f(X)+a,\ a\in R$, and superadditive[1] $f(X+Y)\geq f(X)+f(Y)$. One can show that coherent risk measure f generates a natural shift invariant convex preference relation using (2), and is the certainty equivalent for this relation. Recall that any coherent risk measure may be defined by a family of probability distributions $\mathcal{Q}$ using

$$f(X)=\inf_{Q\in\mathcal{Q}}\mathbf{E}_Q X.$$

Let $|\Omega|=2$, $\mathcal{A}=2^{\Omega}$, так что $\mathcal{X}=R^2$ состоит из точек $X=(X_1,X_2)$. Пусть, далее,

$$\mathbf{P}=\left(\frac{5}{8},\frac{3}{8}\right),\ \mathcal{Q}=\left\{\left(\frac{1}{3},\frac{2}{3}\right),\left(\frac{3}{4},\frac{1}{4}\right)\right\},$$

and a norm on $\mathcal{X}=R^2$ is given by

$$\|X\|=\max_{i=1,2}|X_i|.$$

Then

$$f(X)=\begin{cases}\frac{3}{4}X_1+\frac{1}{4}X_2, & X_1\leq X_2\\ \frac{1}{3}X_1+\frac{2}{3}X_2, & X_1\geq X_2\end{cases}$$

and the set $\mathcal{X}_0$ consists of the points $X=(X_1,X_2)$, satisfying the equation

$$\frac{5}{8}X_1+\frac{3}{8}X_2=0.$$

Unit sphere $\|X\|=1$ intersects the $\mathcal{X}_0$ in two points

$$Y=\left(\frac{3}{5},-1\right),\ Z=\left(-\frac{3}{5},1\right),$$

so

$$r=\max(|f(Y)|,|f(Z)|)=\max\left(\frac{1}{5},\frac{7}{15}\right)=\frac{7}{15}.$$

References

1. J.D. Pratt. Risk Aversion in the Small and in the Large *Econometrics*, **32** (1964), 122-136.
2. K.J. Arrow. *Essays in the Risk Bearing*. Markham, Chicago, 1971.
3. J.H. Dreze. *Essays on Economic Decisions under Uncertainty*. Cambridge University Press, Cambridge, 1987.
4. J. von Neumann, O. Morgenstern. Theory of Games and Economic Behavior. Princeton University Press, Princeton, 1944.
5. Novosyolov A.A. Risk Aversion: A Qualitative Approach and Quantitative Estimates. *Automation and Remote Control*, 64 (2003), 7, 1165-1176.
6. P. Artzner, F. Delbaen, J.-M. Eber. D. Heath. Coherent Measures of Risk. *Mathematical Finance*, 9 (1999), 203-228.
7. Novosyolov A.A. *Mathematical Modeling of Financial Risks*. Nauka: Novosibirsk, 2001.
8. Novosyolov A.A. Representation of Preferences on a Set of Risks by Functional Families. *Proceedings of the 6th International Scientific School "Modeling and Analysis of Safety and Risk in Complex Systems"*. St-Petesrburg, 2006, 92-97.
9. Shiryaev A.N. *Probability*. Nauka: M., 1989.
10. Novosyolov A.A. Generalized Coherent Risk Measures in Decision-Making under Risk. *Proceedings of the 5th International Scientific School "Modeling and Analysis of Safety and Risk in Complex Systems"*. St-Petesrburg, 2005, 145-150.

[1] Strictly speaking in [10] coherent risk measures g were defined in such a way that $g=-f$. In particular super-additivity axiom is replaced by sub-additivity one there.