GENERATORS OF DISTORTED PROBABILITY FUNCTIONALS
ГЕНЕРАТОРЫ ФУНКЦИОНАЛОВ ВОЗМУЩЕННОЙ ВЕРОЯТНОСТИ

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The concept of coherent risk measure is defined axiomatically [1] and every such measure may be represented by a cone of admissible risks or a family of probability distributions [1]. Similar representations are valid for a partial case of coherent risk measures, the so called distorted probability functionals [2, 3]. In the present paper we discuss the specific representation of distorted probability functionals by families of probability measures, and use it to clarify the specific position of distorted probability functionals among coherent risk measures.

Key words: coherent risk measure, distorted probability measure, conditional value at risk, family of probability distributions, generator

Introduction

A class of risk measures called distorted probability measures was introduced in [2, 3]. Later on in [1] the coherent risk measures were defined; they appeared to be generalization of distorted probability measures. A natural question is: how much wider is the new class of risk measures, and what is the specific position of distorted probabilities among coherent functionals. The present paper considers representation of risk measures by families of probability distributions (generators), which allows in some sense answering the question.

The paper is organized as follows: the next section is devoted to definitions of risk measures and establishing the relation between coherent functionals and distorted probabilities. Then a separate section describes representation of risk measures by families of probability distributions (generators); in particular it contains the main result on such a representation of distorted probability functionals. The final section contains an example of generator for distorted probability functional.

Coherent risk measures

Let $(\Omega, \mathbf{A}, \mathbf{P})$ be a probability space, $(\mathbf{R}, \mathbf{B})$ be a real measurable space with Borel σ-algebra $\mathbf{B}$, and $\mathbf{X}$ be a set of random variables, i.e. measurable mappings $X : \Omega \to \mathbf{R}$. The set $\mathbf{X}$ forms a linear space with respect to natural operations with functions. Each random variable $X \in \mathbf{X}$ generates the distribution P_X on $(\mathbf{R}, \mathbf{B})$ according to the rule

$$P_X(B) = \mathbf{P}\left(X^{-1}(B)\right), \quad B \in \mathbf{B}.$$

This distribution may be characterized by the distribution function

$$F_X(x) = P_X\left((-\infty, x]\right), \quad x \in \mathbf{R}.$$

Different random variables $X, Y \in \mathbf{X}$, generally speaking, may generate equal distributions $P_X = P_Y$.

We call *risk measure* any functional $f : \mathbf{X} \to \mathbf{R}$. A risk measure f is called *regular* if $f(X) = f(Y)$ for any random variables $X, Y \in \mathbf{X}$ possessing the property $P_X = P_Y$. Denote $I \equiv 1$ the random variable identically equal to 1, and describe a few properties that risk measures may possess.

(**M**) monotonicity $X \le Y \Rightarrow f(X) \le f(Y)$;
(**H**) positive homogeneity $f(\lambda X) = \lambda f(X)$;
(**I**) translation invariance $F(X + aI) = f(X) + a$;
(**S**) super-additivity $f(X + Y) \ge f(X) + f(Y)$.

A risk measure f is called *coherent*[1], if it possesses the properties (**M**), (**H**), (**I**), (**S**).

Distorted probability measure is defined [2, 3] as follows:

$$\pi_g(X) = \int_{-\infty}^{0} [g(1 - F_X(t)) - 1]dt + \int_{0}^{\infty} g(1 - F_X(t))dt, \qquad X \in \mathbf{X}; \tag{1}$$

here the function $g:[0,1] \to [0,1]$ is non-decreasing and satisfies $g(0) = 0$, $g(1) = 1$. The latter function is a parameter of distorted probability functional.

Theorem 1. A functional π_g is a regular risk measure. If the function g is convex then π_g is a coherent risk measure.

Proof. Regularity of the distorted probability functional is evident from its definition (1). Properties (**M**), (**H**), (**I**) are easily checked. Besides that in [4] it was shown that convexity of g is sufficient for super-additivity of the functional π_g. Thus the functional π_g with convex parameter g is a coherent risk measure, as required. ◊

Representation by families of probability measures

Let Q be a probability measure on a measurable space $(\Omega, \mathbf{A})$. For $X \in \mathbf{X}$ denote

$$\mathbf{E}_Q X = \int_{\Omega} X(\omega) dQ(\omega)$$

the expectation of X with respect to Q. In [1] the following theorem was presented.

Theorem 2. Let f be a coherent risk measure. Then there is a family of probability measures $\mathbf{Q} = \mathbf{Q}(f)$ on $(\Omega, \mathbf{A})$ such that

$$f(X) = \inf_{Q \in \mathbf{Q}} \mathbf{E}_Q X. \Diamond \tag{2}$$

Note that the inverse theorem is obviously true. We will call *generator* of a coherent risk measure f the family $\mathbf{Q}$ appearing in (2). Note also that together with the family $\mathbf{Q}$, its convex hull $Co(\mathbf{Q})$, and each family $\tilde{\mathbf{Q}}$ in between $\mathbf{Q} \subseteq \tilde{\mathbf{Q}} \subseteq Co(\mathbf{Q})$, may also play a role of generator for f. In particular, under the conditions of Krein – Milman theorem [5], the smallest (by inclusion) generator $\mathbf{Q}_m(f)$ consists of extreme points of some generator $\mathbf{Q}(f)$.

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[1] Strictly speaking, in [1] the term "coherent risk measures" is associated with functionals $(-f)$, yet all results are easily transferred from one treatment to another.

Now let us obtain more precise representation (2) for the case of distorted probability functional π_g. To avoid technical difficulties we will consider only the finite case $|\Omega| = n$. In this case the main probability measure is described by an n-taple of probabilities $P = (p_1, p_2, \ldots, p_n)$, the generic probability measure Q on $(\Omega, \mathbf{A})$ is represented by a vector of probabilities $Q = (q_1, q_2, \ldots, q_n)$, and any random variable $X \in \mathbf{X}$ is completely described by a vector $X = (x_1, x_2, \ldots, x_n)$.

From [4] we know that if $X = (x_1, x_2, \ldots, x_n)$ such that

$$x_1 \le x_2 \le \cdots \le x_n, \tag{3}$$

then the value of π_g at X is calculated by

$$\pi_g(X) = \sum_{k=1}^{n} x_k q_k = \mathbf{E}_Q X, \tag{4}$$

where

$$q_k = g(r_k) - g(r_{k+1}), \quad r_k = \sum_{i=k}^{n} p_i, \quad k = 1, \ldots, n, \quad r_{n+1} = 0.$$

Clearly $q_k \ge 0$, $k = 1, \ldots, n$, and $q_1 + \cdots + q_n = 1$, so $Q = (q_1, \ldots, q_n)$ is actually a probability measure, which justifies using the expectation notation in (4).

Now let $X \in \mathbf{X}$ be such that (3) does not hold. Consider a permutation γ of the set $\mathbf{N} = \{1, \ldots, n\}$ (that is, a mapping of $\mathbf{N}$ onto itself) such that inequalities

$$x_{\gamma(1)} \le x_{\gamma(2)} \le \cdots \le x_{\gamma(n)} \tag{5}$$

hold. Then the value of π_g at X is calculated as

$$\pi_g(X) = \sum_{k=1}^{n} x_k q_k^\gamma = \mathbf{E}_{Q^\gamma} X, \tag{6}$$

where

$$Q^\gamma = (q_1^\gamma, \ldots, q_n^\gamma), \qquad q_k^\gamma = g(r_k^\gamma) - g(r_{k+1}^\gamma), \quad r_k^\gamma = \sum_{i=k}^{n} p_{\gamma(i)}, \quad k = 1, \ldots, n, \quad r_{n+1}^\gamma = 0. \tag{7}$$

Now we can conclude that to calculate a value of π_g at an arbitrary point $X \in \mathbf{X}$ it suffices to know $n!$ probability measures Q^γ for all permutations γ of the set $\mathbf{N}$. Denote Γ this set of all permutations. Obviously, by construction, the minimum value of $\mathbf{E}_{Q^\gamma} X$ is attained for the permutation γ satisfying (5). Denoting $\mathbf{Q} = \{Q^\gamma, \gamma \in \Gamma\}$, we get

$$\pi_g(X) = \min_{Q \in \mathbf{Q}} \mathbf{E}_Q X. \tag{8}$$

Thus we have obtained the following theorem.

Theorem 3. Let the distortion function g be convex, and $|\Omega| = n$. Then the generator for the distortion probability functional π_g has the form $\mathbf{Q} = \{Q^\gamma, \gamma \in \Gamma\}$, where Q^γ are defined in (7); in other words, the functional π_g is defined by (8). As noted before, the generator may be enlarged up to the convex hull of $\mathbf{Q}$, which is a convex polyhedron in the standard simplex of probability measures. ◊

Example

Let $|\Omega| = 3$, $\mathbf{P} = (1/4, 1/4, 1/2)$, $g(v) = v^2$, $v \in [0,1]$. The set of all probability measures form a standard simplex in $\mathbf{R}^3$, which is an equilateral triangle. In Table 1 we present all the permutations γ of the set $\mathbf{N} = \{1,2,3\}$, the corresponding probability measures Q^γ, and the values of expectations of $X = (1,5,3)$ with respect to Q^γ. One can see that the permutation leading to ordered coordinates is $\gamma = (1,3,2)$, and the minimum expectation 36/16 is placed in the corresponding row of the table.

γ	Q^γ	$\mathbf{E}_{Q^\gamma} X$
(1,2,3)	(7/16,5/16,1/4)	44/16
(1,3,2)	(7/16,1/16,1/2)	36/16
(2,1,3)	(5/16,7/16,1/4)	52/16
(2,3,1)	(1/16,7/16,1/2)	60/16
(3,1,2)	(3/16,1/16,3/4)	44/16
(3,2,1)	(1/16,3/16,3/4)	52/16

Table 1. Generator $\mathbf{Q}$ and expectations for $X = (1,5,3)$

Figure 1 presents a polygon whose vertices are in $\mathbf{Q}$, in the plane of the standard simplex of $\mathbf{R}^3$. In the general case of finite Ω the generator has the form of polyhedron with boundary hyper planes parallel to the boundaries of the standard simplex.

Note that generator of a coherent risk measure may have the form of any convex subset of the standard simplex (or even the set of its extreme points and any set in between, as noted above).

Conclusion

In the present paper we have considered representation of coherent risk measures and there partial case, distorted probability measures, by families of probability measures. Explicit form of generators for distorted probability measures in case of finite probability spaces have been obtained; this makes clear the difference between this partial case and the general coherent case.

Results of the paper may prove useful in deriving properties of distorted probability functional, and especially, in constructing algorithms for fast computing. It is worth noting that conditional value at risk (CVaR) functional, which gains much attention in the literature and in applications, is a special case of distorted probability functional. Deriving specific representations and calculation algorithms for CVaR is a matter of further research.

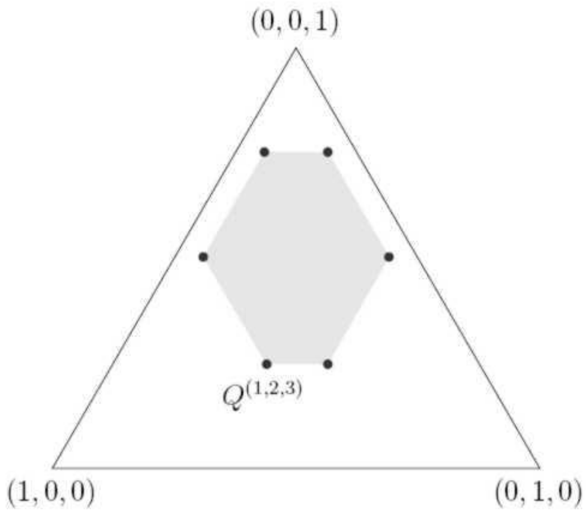

Figure 1. Generator of a distorted probability functional

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