COPULA FUNCTION AS A DEPENDENCE STRUCTURE

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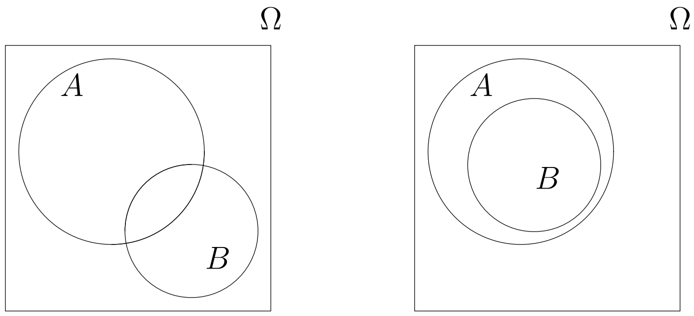
Abstract

Correlation structure of a multidimensional probability distribution captures only a small fraction of information on components dependence. To overcome the lack of information copula functions were intensively used during recent decades. The present paper gives an overview of copula functions as descriptors of dependence structure, presents some general classes of copula functions, and provides algorithms for using copula functions in decision-making under risk. Special attention is paid to discrete marginals case, which has not been addressed much in the literature before.

Frechet bounds

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, $A, B \in \mathcal{A}$, $\mathbf{P}(A) = p$, $\mathbf{P}(B) = q$.

$$\max_{\mathbf{P}(A)=p, \mathbf{P}(B)=q} \mathbf{P}(A \cap B) = ? \qquad \min_{\mathbf{P}(A)=p, \mathbf{P}(B)=q} \mathbf{P}(A \cap B) = ?$$

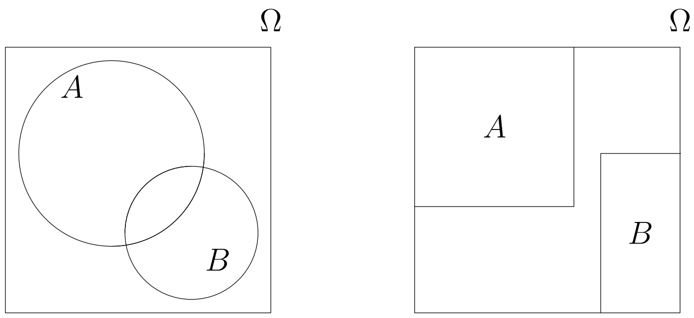


a) general position b) maximum $\mathbf{P}(A \cap B)$

Figure 1: Maximum intersection probability

Upper Frechet bound

$$\max_{\mathbf{P}(A)=p, \mathbf{P}(B)=q} \mathbf{P}(A \cap B) = \min\{p, q\}$$



a) general position b) maximum $\mathbf{P}(A \cap B)$

Figure 2: Minimum intersection probability

Lower Frechet bound

$$\min_{\mathbf{P}(A)=p, \mathbf{P}(B)=q} \mathbf{P}(A \cap B) = \max\{0, p + q - 1\}$$

Examples

$$p = 0.5, \; q = 0.5, \qquad 0 \le \mathbf{P}(A \cap B) \le 0.5$$
$$p = 0.6, \; q = 0.7, \qquad 0.3 \le \mathbf{P}(A \cap B) \le 0.6$$

General Frechet bounds (n events)

$$A_1, \dots, A_n \in \mathcal{A} \implies$$
$$\max\left\{0, \sum_{i=1}^{n} \mathbf{P}(A_i) - n + 1\right\} \le \mathbf{P}\left(\bigcap_{i=1}^{n} A_i\right) \le \min\left\{\mathbf{P}(A_1), \dots, \mathbf{P}(A_n)\right\}$$

Distribution functions
Random vector $(X_1, \dots, X_n)$;

$$F_{X_1,\dots,X_n}(x_1, \dots, x_n) = \mathbf{P}(X_1 \le x_1, \dots, X_n \le x_n)$$
$$F_{X_i}(x_i) = \mathbf{P}(X_i \le x_i), \;\; i = 1, \dots, n; \;\; x = (x_1, \dots, x_n) \in \mathbf{R}^n$$

Frechet bounds for distribution functions

$$\max\left\{0, \sum_{i=1}^{n} F_{X_i}(x_i) - n + 1\right\} \le F_{X_1,\dots,X_n}(x_1, \dots, x_n) \le$$
$$\le \min\left\{F_{X_1}(x_1), \dots, F_{X_n}(x_n))\right\}; \;\; x = (x_1, \dots, x_n) \in \mathbf{R}^n$$

Transform
Define
$$U_1 = F_{X_1}(X_1), \dots, U_n = F_{X_n}(X_n).$$
$U_1, \dots, U_n$ are uniform over $[0, 1]$. What is the distribution function of $(U_1, \dots, U_n)$? Here we go for $(u_1, \dots, u_n) \in [0, 1]^n$:

$$\mathbf{P}(U_1 \le u_1, \dots, U_n \le u_n) = \mathbf{P}(F_{X_1}(X_1) \le u_1, \dots, F_{X_n}(X_n) \le u_n)$$
$$= \mathbf{P}(X_1 \le F_{X_1}^{-1}(u_1), \dots, X_n \le F_{X_n}^{-1}(u_n)) = F_{X_1,\dots,X_n}(F_{X_1}^{-1}(u_1), \dots, F_{X_n}^{-1}(u_n))$$

This joint distribution function for $(U_1, \dots, U_n)$ is called copula function.

Inverse transform

$$X_1 = F_{X_1}^{-1}(U_1), \dots, X_n = F_{X_n}^{-1}(U_n).$$

Copula function

$$C(u_1, \ldots, u_n) = F_{X_1,\ldots,X_n}(F_{X_1}^{-1}(u_1), \ldots, F_{X_n}^{-1}(u_n)), \ \ (u_1, \ldots, u_n) \in [0,1]^n$$

C is the distribution function for $(U_1, \ldots, U_n)$ with $U_i = F_{X_i}(X_i), \ i = 1, \ldots, n$

Distribution function representation via copula

$$F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = C\left(F_{X_1}(x_1), \ldots, F_{X_n}(x_n)\right); \ \ (x_1, \ldots, x_n) \in \mathbf{R}^n \quad (1)$$

Independent definition

Copula is a distribution function on $[0,1]^n$ with uniform marginals

Sklar's theorem (1959)
For any distribution function $F_{X_1,\ldots,X_n}$ there exists a copula function C such that a representation (1) is valid. If marginal distribution functions $F_{X_1}, \ldots, F_{X_n}$ are continuous, then the representation (1) is unique.

Denote $\mathcal{C}^n$ the collection of all copula functions for a given n.

Frechet class
Given a collection of one-dimensional distribution functions $F_1, \ldots, F_n$, the Frechet class $\mathcal{F}(F_1, \ldots, F_n)$ is the set of all distribution functions F on $\mathbf{R}^n$ having $F_1, \ldots, F_n$ as its marginals, i.e.

$$F(x_1, \infty, \ldots, \infty) = F_1(x_1), \ x_1 \in \mathbf{R}, \ \ \ldots$$

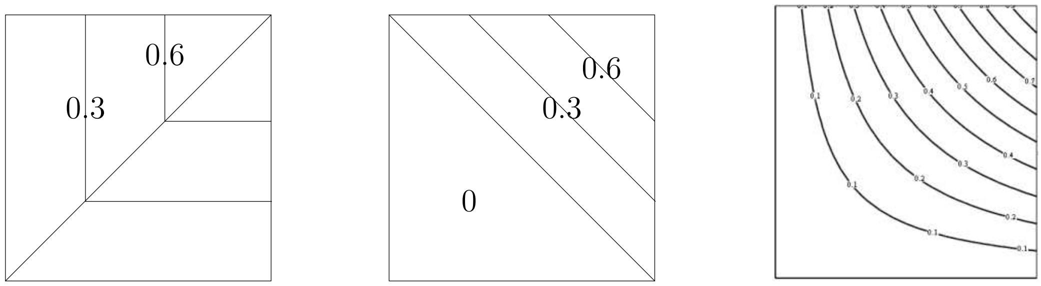
Frechet class may be obtained via

$$\mathcal{F}(F_1, \ldots, F_n) = \{F : \ F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)), \ C \in \mathcal{C}^n\}.$$

In particular, $\mathcal{C}^n = \mathcal{F}(G, \ldots, G)$, where G is the uniform [0,1] distribution function.

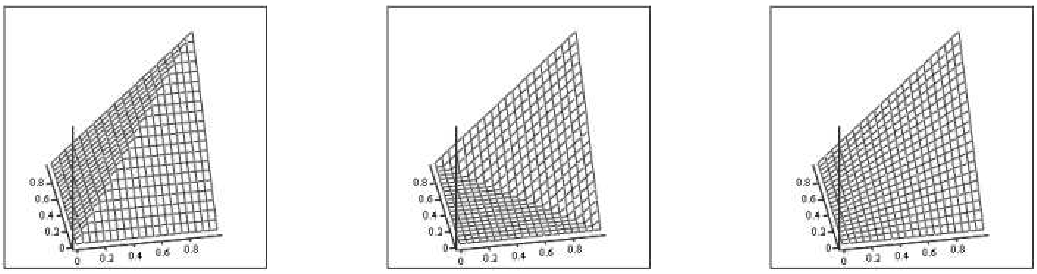
Simple copulas

- Upper Frechet bound $C^{+}(u_1,\ldots,u_n) = \min\{u_1,\ldots,u_n\}$
- Lower Frechet bound $C^{-}(u_1,\ldots,u_n) = \max\{0, \sum_{i=1}^{n} u_i - n + 1\}$ (copula for $n \leq 2$)
- Independent copula $C^{\perp}(u_1,\ldots,u_n) = u_1 \cdots u_n$.



a) C^{+} b) C^{-} c) $C^{\perp}$

Figure 3: Level curves for simple copulas



a) C^{+}, $U_1 = U_2$ b) C^{-}, $U_1 = 1 - U_2$ c) $C^{\perp}$

Figure 4: Graphs of simple copulas

Comonotone variables

Components of $(X_1,\ldots,X_n)$ are called comonotone if the corresponding copula is C^{+}

Anticomonotone variables ($n = 2$)

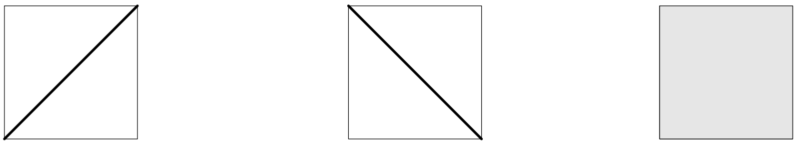
Components of (X_1, X_2) are called anticomonotone if the corresponding copula is C^{-}

Density copula

$$c(u_1, \ldots, u_n) = \frac{\partial^n C(u_1, \ldots, u_n)}{\partial u_1 \cdots \partial u_n}$$

Density representation

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = c\left(F_{X_1}(x_1), \ldots, F_{X_n}(x_n)\right) \prod_{i=1}^{n} f_{X_i}(x_i)$$



a) c^+, $U_1 = U_2$ b) c^-, $U_1 = 1 - U_2$ c) $c^\perp$

Figure 5: "Densities" for simple copulas

Example: Cook-Johnson copulas

$$C_\alpha(u_1, \ldots, u_n) = \left(\sum_{i=1}^{n} u_i^{-1/\alpha} - n + 1 \right)^{-\alpha}, \ \alpha > 0$$

$$C_0(u_1, \ldots, u_n) = C^+(u_1, \ldots, u_n); \ \ C_\infty(u_1, \ldots, u_n) = C^\perp(u_1, \ldots, u_n);$$

Archimedean copulas

Generator $g : (0, 1] \to [0, \infty)$, $g(1) = 0$, $g'(t) < 0$, $g''(t) > 0$, $t \in (0, 1]$.

$$C(u_1, \ldots, u_n) = C_g(u_1, \ldots, u_n) =$$

$$= \begin{cases} g^{-1}\left(g(u_1) + \cdots + g(u_n)\right), & g(u_1) + \cdots + g(u_n) \le g(0) \\ 0, & \text{otherwise} \end{cases}$$

Examples

Class	Generator g	Copula $C_g(u, v)$
$C^\perp$	$-\ln u$	uv
C^-	$1 - u$	$\max(0, u + v - 1)$
Gumbel	$(-\ln u)^\alpha$	$\exp\left(-\left[g(u) + g(v)\right]^{1/\alpha}\right)$

Normal copula

$$C_S(u_1,\ldots,u_n) = \Phi_S\left(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_n)\right)$$

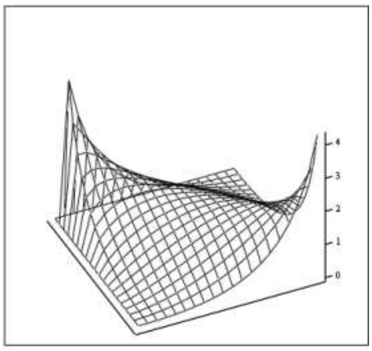
$$\varphi_S(x) = \frac{1}{(2\pi)^{n/2}|S|^{1/2}}\ \exp\left(-\frac{1}{2}x^T S^{-1} x\right); \qquad S = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{12} & 1 & \cdots & \rho_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{1n} & \rho_{2n} & \cdots & 1 \end{pmatrix}$$

2-dimensional case

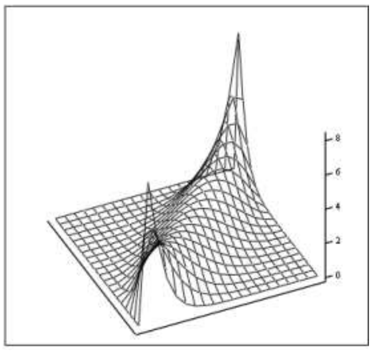
$$S = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \quad \varphi(t) = \frac{1}{2\pi}\exp\left(-\frac{t^2}{2}\right), \quad \Phi(w) = \int_{-\infty}^{w}\varphi(t)\,dt$$

$$\varphi_S(x_1,x_2) = \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left(-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}\right)$$

$$c(u_1,u_2) = \frac{\varphi_S(\Phi^{-1}(u_1),\Phi^{-1}(u_2))}{\varphi(\Phi^{-1}(u_1))\varphi(\Phi^{-1}(u_2))}$$



a) $\rho = 0.7$



b) $\rho = -0.9$

Figure 6: Normal density copulas

Special cases

- $\rho = 0 \implies$ independent case, $C^{\perp}$, $c^{\perp}(u_1,u_2) \equiv 1$
- $\rho \to 1 \implies C_S \to C^{+}$, upper Frechet bound
- $\rho \to -1 \implies C_S \to C^{-}$, lower Frechet bound

Monte Carlo simulation

Given a copula $C(u_1, \ldots, u_n)$, $(u_1, \ldots, u_n) \in [0,1]^n$ and marginal distributions $F_1, \ldots, F_n$. How to generate the random vector $(X_1, \ldots, X_n)$ with the distribution function

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)) \quad ?$$

2-stage algorithm

1) Generate a random vector $(U_1, \ldots, U_n)$ with distribution function C
2) Make the transform $X_1 = F_1^{-1}(U_1), \ldots, X_n = F_n^{-1}(U_n)$

Specific methods

Upper Frechet bound

A) Generate U uniform(0,1)
B) Set $U_1 = \ldots = U_n = U$

Lower Frechet bound ($n = 2$)

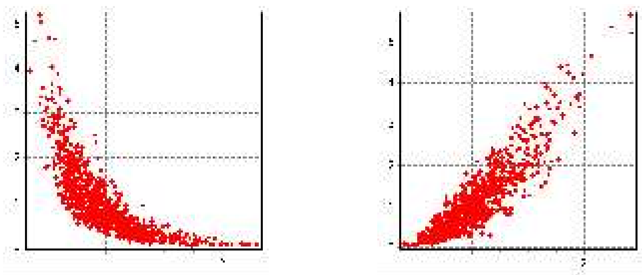
A) Generate U uniform(0,1)
B) Set $U_1 = U, \ U_2 = 1 - U$

Cook-Johnson copula

A) Generate $Y_1, \ldots, Y_n$ independent exponential(1)
B) Generate $Z \sim$ Gamma(α,1)
C) Transform $U_i = (1 + Y_i/Z)^{-\alpha}, \ i = 1, \ldots, n$

Normal copula with correlation matrix S

A) Calculate Cholesky decomposition $S = AA^T$
B) Generate $Y_1, \ldots, Y_n$ independent standard normal
C) Calculate $Z = AY$
D) Transform $U_i = \Phi(Z_i), \ i = 1, \ldots, n$



a) $\rho = -0.9$ b) $\rho = 0.9$

Figure 7: Simulation, normal copulas, lognormal+gamma marginals

Canonic copula for discrete distribution

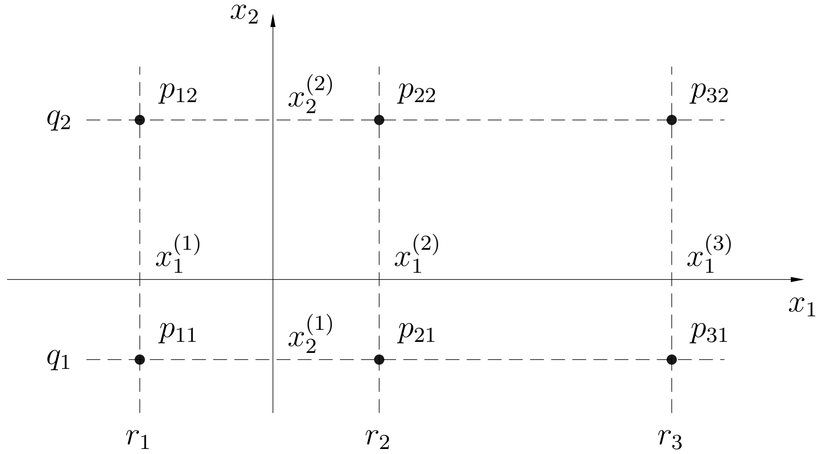


Figure 8: Discrete distribution

$$\text{Joint distribution: } p_{ij} \geq 0;\ i = 1, 2, 3;\ j = 1, 2; \quad \sum_{i,j} p_{ij} = 1$$

$$\text{Marginal distributions: } r_i = \sum_{j=1}^{2} p_{ij},\ i = 1, 2, 3; \ \ q_j = \sum_{i=1}^{3} p_{ij},\ j = 1, 2$$

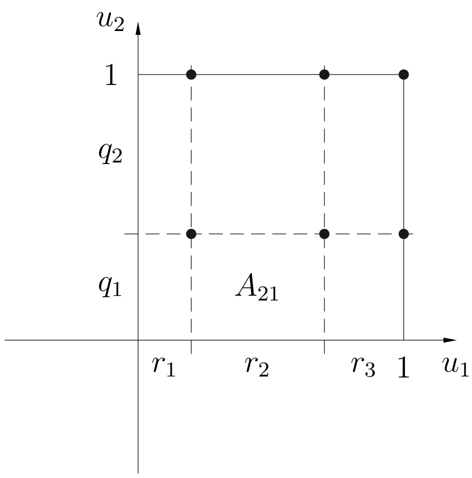


Figure 9: Canonic copula

$$u = (u_1, u_2) \in A_{ij} \iff \sum_{k=1}^{i-1} r_k < u_1 \leq \sum_{k=1}^{i} r_k, \ \ \sum_{s=1}^{j-1} q_s < u_2 \leq \sum_{s=1}^{j} q_s$$

$$c(u_1, u_2) = \frac{p_{ij}}{r_i q_j},\ u \in A_{ij}$$

Some classes of canonic copulas

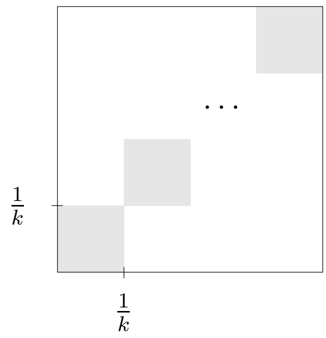


Figure 10: Positive correlation copulas C_k

Correlation $\rho_k = 1 - \frac{1}{k^2}$

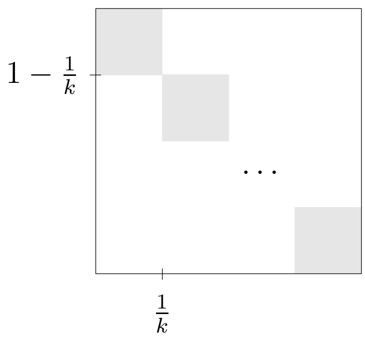


Figure 11: Negative correlation copulas C_k

Correlation $\rho_k = -1 + \frac{1}{k^2}$

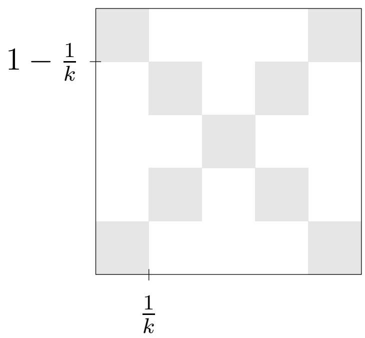


Figure 12: Combined dependence copulas C_k ($k = 5$ here)

Correlation $\rho_k = 0$

Example: Correlation variability under nonlinear increasing transforms

Y value	Prob			
1	1/2	0	1/2	0
0	1/2	1/4	0	1/4
Prob		1/4	1/2	1/4
X_a value		0	a	2

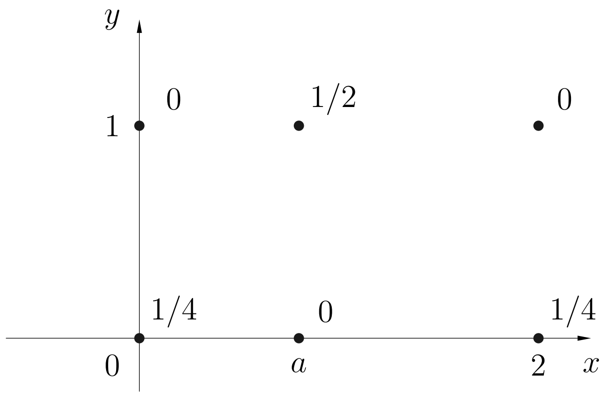


Figure 13: Discrete distribution

$$\mathbf{E}X_a = \frac{a+1}{2}\,,\ \mathbf{D}X_a = \frac{a^2-2a+3}{4}\,,\ \mathbf{E}Y = \frac{1}{2}\,,\ \mathbf{D}Y = \frac{1}{4}\,,\ \mathbf{E}(X_aY) = \frac{a}{2}$$

$$\boxed{Cov(X_a,Y) = \tfrac{a-1}{4}\,,\ Corr(X_a,Y) = \tfrac{a-1}{\sqrt{a^2-2a+3}}}$$

$$\boxed{0 < a < 2 \implies -\tfrac{1}{\sqrt{3}} \le Corr(X_a,Y) \le \tfrac{1}{\sqrt{3}}}$$

In other words, $Cov(X_1,Y) = 0$ and $-\frac{1}{\sqrt{3}} \le Cov(f(X_1),Y) \le \frac{1}{\sqrt{3}}$, where $f:\,[0,2] \to [0,2]$ is strictly monotone, e.g.

$$f(x) = 2\left(\frac{1}{2}\,x\right)^{\beta},\ x \in [0,2] \qquad (\beta > 0)$$

Example: negative correlation in dimension 3

Normal vector $X = (X_1, X_2, X_3)^T$ with means $m = (0,0,0)^T$ and covariance matrix

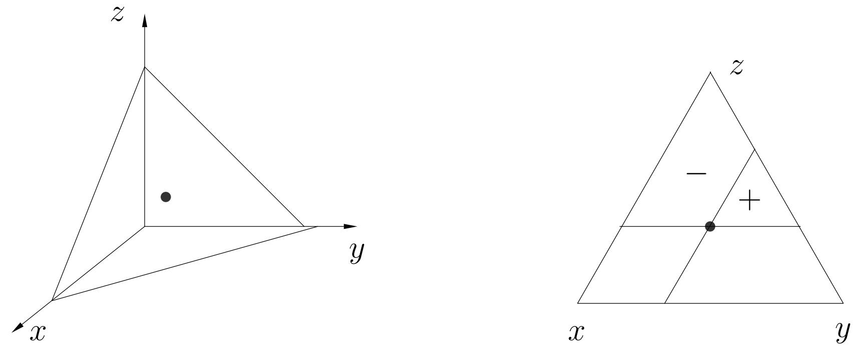
$$\mathbf{E}\left(XX^T\right) = V = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}$$

V is positive definite for $\rho > -1/2$.

Limiting case $\rho = -1/2$. Here

$$V = \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix},$$

$|V| = 0$, $rank(V) = 2$, distribution of X is degenerate normal, restricted to the plane $x_1 + x_2 + x_3 = 0$, $x = (x_1, x_2, x_3)^T \in \mathbf{R}^3$. Consider a shift by $(1/3, 1/3, 1/3)^T$, now the distribution with means $m = (1/3, 1/3, 1/3)^T$ is restricted to the plane $x_1 + x_2 + x_3 = 1$.



a) in $\mathbf{R}^3$ b) projection to $x_1 + x_2 + x_3 = 1$

Figure 14: Standard simplex