REPRESENTATION OF PREFERENCES ON A SET OF RISKS BY FUNCTIONAL FAMILIES

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Decision-making under risk

$\mathcal{D}$ — set of decisions; $\mathcal{R}$ — set of results

$\mathcal{F}$ — set of probability distributions over R

$m : \mathcal{D} \to \mathcal{F}$ — decision model

$d \in \mathcal{D} \implies F = m(d) \in \mathcal{F}$

Best decision provides **best** (in some sense) distribution

The best = most **preferable**

Establishing preference

- Calculating expectation
- Calculating standard deviation
- Calculating Value-at-Risk
- Calculating Conditional Value-at-Risk
- Calculating expected utility
- Calculating other functionals

Preference by functional

Given some functional $f : \mathcal{F} \to \mathbf{R}$

$$F, G \in \mathcal{F}, \quad f(F) \leq f(G) \implies F \preceq G$$

Example: expected utility

Given $U : \mathbf{R} \to \mathbf{R}$

$$f(F) = f_U(F) = \int_{-\infty}^{\infty} U(x)\, dF(x), \quad F \in \mathcal{F}$$

Functional by preference: representation

Given a preference relation $\preceq$ on $\mathcal{F}$

Find a functional $f : \mathcal{F} \to \mathbf{R}$ such that

$$F, G \in \mathcal{F}, \quad F \preceq G \implies f(F) \leq f(G)$$

Classics: expected utility

Theorem (von Neumann, Morgenstern, 1944). If the preference relation is linear and continuous, then there exists a representing functional (expected utility). The functional is unique up to positive affine transforms.

Disadvantage: linearity assumption often leads to quite rough models

Hence: there's a strong need in finding representing functionals under more realistic assumptions

Simple additional assumptions

Decision-maker is a money lover (the more, the better). Thus the preference relation is concordant with **stochastic dominance**. In **expected utility** model the assumption implies **monotonicity** of the utility function.

Decision-maker is risk averse. The assumption restricts the set of admissible preference relations. E.g., in **expected utility** model the assumption implies **concavity** of the utility function.

Some classes of representing functionals

- Coherent risk measures (Artzner et al, 1999)
- Convex risk measures (Follmer, Schied, 2002)
- Generalized coherent risk measures (Jarrow, Purnanandam, 2005, Novosyolov, 2005)
- Quantile risk measures (VaR, CVaR, back to ancient times)

Problems statement

- Given a preference relation on a set of probability distributions; find a representing functional
- Given a partial preference relation on a set of probability distributions (risks); find a family of representing functionals

Basic assumptions

- $\mathcal{F}$ is the set of all real distributions with bounded support
- all (complete and partial) preference relations are continuations of the stochastic dominance order on $\mathcal{F}$
- all preference relations are upper and lower semicontinuous

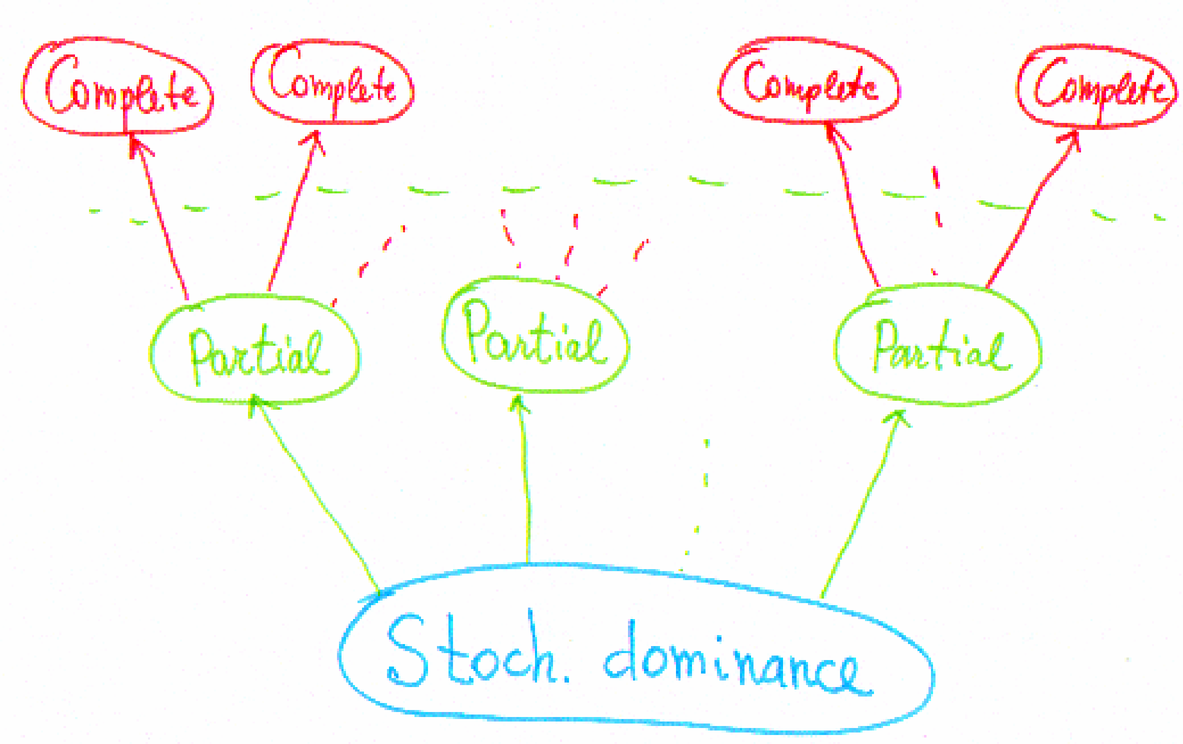
Preference representation

Theorem 1. Let a complete preference relation on the set of risks be regular. Then there exists the canonical representing functional.

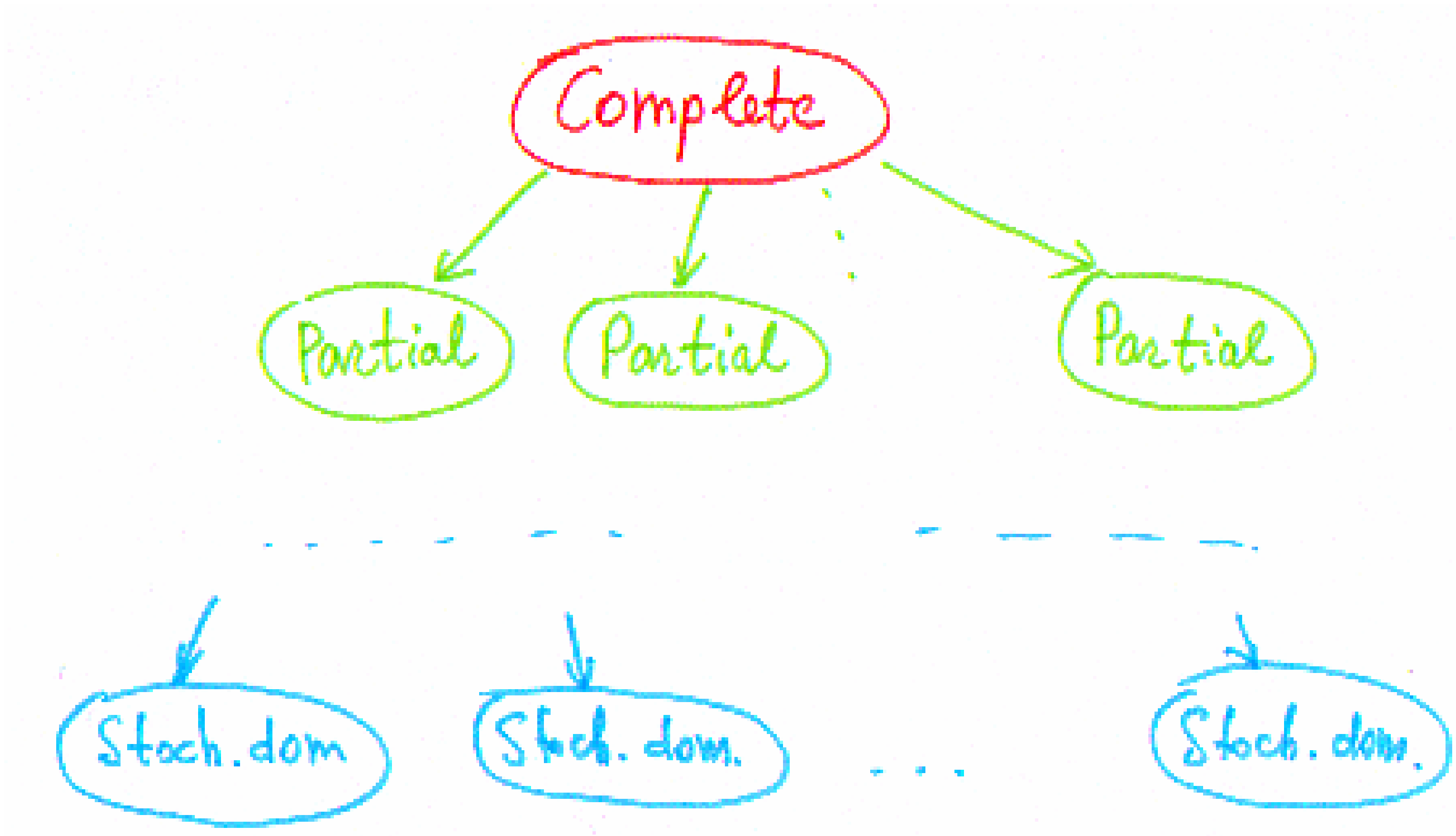
Theorem 2. Let a regular partial preference relation be a restriction of a regular complete preference on the set of risks. Then there exists a representing family of canonical functionals.

Remark. Theorem 2 implicitly contains a method for sequential approximation of a complete preference by partial ones.

Preference relations tree



Functional families tree



Appendix

math details

Binary relations

Binary relation Q on a set $\mathcal{F} \longleftrightarrow$ any subset $Q \subseteq \mathcal{F} \times \mathcal{F}$

Example: *diagonal* $\mathcal{I}_{\mathcal{F}} = \{(x,x),\ x \in \mathcal{F}\}$ (*equality* relation), shown below for $\mathcal{F} = \{a,b,c\}$

	a	b	c
a	●	○	○
b	○	●	○
c	○	○	●

Operations with binary relations

Transposition: $Q^T = \{(y, x) : (x, y) \in Q\}$
Composition:

$$Q \circ R = \{(x, y) : \exists z \in \mathcal{F}, (x, z) \in Q, (z, y) \in R\}$$

Symmetric part: $Q^s = Q \cap Q^T$
Asymmetric part: $Q^a = Q \setminus Q^s$

Properties of binary relations

(**R**) *Reflexive*: $Q \supseteq \mathcal{I}_{\mathcal{F}}$
(**S**) *Symmetric*: $Q = Q^T$
(**T**) *Transitive*: $Q \circ Q \subseteq Q$
(**A**) *Antisymmetric*: $Q \cap Q^T \subseteq \mathcal{I}_{\mathcal{F}}$
(**C**) *Complete*: $Q \cup Q^T = \mathcal{F} \times \mathcal{F}$

Specific relations

Equivalence: $(\mathbf{R\ S\ T})$; notation $\sim$

Order: $(\mathbf{R\ A\ T})$; notation $\leq$

Preference: $(\mathbf{T\ C})$; notation $\preceq$

Partial preference: $(\mathbf{T\ R})$; notation $\preceq$

Continuation (restriction), $Q, R \subseteq \mathcal{F} \times \mathcal{F}$

$Q^s \subseteq R^s, \quad Q^a \subseteq R^a$

Probability distributions

Distribution function $F(x) = \mathbf{P}((-\infty, x]), \quad x \in \mathbf{R}$

Distribution with bounded support

$$F(a) = 0, \; F(b) = 1 \text{ for some } -\infty < a < b < \infty$$

Degenerate distribution

$$W_a(x) = \begin{cases} 0, & x < a \\ 1, & x \geq a \end{cases}$$

Stochastic dominance

Stochastic dominance is a partial order on $\mathcal{F}$

$F \leq G \iff F(x) \geq G(x), \; x \in \mathbf{R}$

Representing functionals

Canonic representing functional $f(W_a) = a, \ a \in \mathbf{R}$

Representing family of functionals $\mathcal{M}$

$Q_f^s = \{(F, G) : \ f(F) = f(G)\}$

$Q_f^a = \{(F, G) : \ f(F) < f(G)\}$

$$Q_{\mathcal{M}}^s = \bigcap_{f \in \mathcal{M}} Q_f^s, \quad Q_{\mathcal{M}}^a = \bigcap_{f \in \mathcal{M}} Q_f^a, \quad Q_{\mathcal{M}} = Q_{\mathcal{M}}^s \cup Q_{\mathcal{M}}^a$$