# INVERSE PROBLEMS OF RISK THEORY AND CHARACTERISTIC CLASSES OF DISTRIBUTIONS

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**Abstract**

Building a model of individual preferences is a key for rational decision-making under uncertainty. Solution of this inverse problem may be simplified by proper using of available information. The present paper introduces the concept of characteristic class of a family of preferences, and presents usage of the concept for solving inverse problems. Characteristic classes for a number of families of preferences have been calculated.

**Keywords**

Decision-making, risk, individual preferences, risk measure, expected utility, distorted probability, combined functional, characteristic class

## Introduction

Decision-making under risk usually reduces to choosing the best probability distribution in a predefined set of distributions [1], where "the best" means the most preferable distribution in the sense of individual preferences. The preferences may often be represented by a real functional [1, 2] called risk measure; a classic representation of the sort is the representation of linear preferences by expected utility functional [3].

Since preferences differ for different decision-makers, it may be a challenging problem to build a reasonable preference for a specific problem, or, equivalently, to find a representing risk measure. This problem of building a preference relation or representing risk measure may be thought of as an inverse problem of risk theory.

Simplifying of inverse problems may be achieved by restricting them to a small class of distributions, if possible. Such simplifying occurs possible in presence of additional information about preference relation (risk measure). Specifically, if the preference relation is known to belong to a family of preference relations, then the corresponding inverse problem is to be solved on a proper subset of the initial domain, which is called a characteristic class of the family. Determining the preference on a characteristic class is sufficient for reconstructing the complete preference; in other words, the continuation of a preference from a characteristic class to the whole domain is unique within the family specified.

In the present paper we describe the concept of characteristic class and calculate characteristic classes for a number of families of risk measures and preferences.

## Characteristic classes

Let $\mathbf{D}$ be a set of acts (decisions) and $\mathbf{F}$ be a set of probability distribution functions on the real line $\mathbf{R}$. Each decision $d \in \mathbf{D}$ provides a distribution function $F \in \mathbf{F}$, the latter representing an uncertain (risky) outcome of a decision $d$. Let $f : \mathbf{D} \to \mathbf{F}$ be a mapping describing this correspondence, that is, $F = f(d)$ is a distribution function corresponding to $d$.

Let also $\prec=$ be a preference relation on $\mathbf{F}$, that is, a complete transitive relation on $\mathbf{F}$, which is treated as follows: $F \prec= G$ means that the distribution $G \in \mathbf{F}$ is at least as good as the distribution $F \in \mathbf{F}$, perhaps even better. Alternatively preference relation $\prec=$ may be thought of as a subset $P \subseteq \mathbf{F} \times \mathbf{F}$ of a Cartesian product of $\mathbf{F}$ by itself with clear treatment "$F \prec= G$ if and only if $(F,G) \in P$". The decision-making problem essentially consists of choosing a decision $d^* \in \mathbf{D}$ such that $f(d) \prec= f(d^*)$ for all $d \in \mathbf{D}$. We will also use the notation $F \prec G$ and $F \sim G$ for asymmetric and symmetric parts of $\prec=$, respectively.

As it was shown in [1], a preference relation $\prec=$ may be represented by a real functional (risk measure) $\mu : \mathbf{F} \to \mathbf{R}$ in the sense

$$F \prec= G \quad \Leftrightarrow \quad \mu(F) \le \mu(G), \quad F,G \in \mathbf{F}. \tag{1}$$

In terms of risk measure the decision-making problem may be reformulated as a traditional optimization problem

$$\mu(f(d)) \to \max_{d \in \mathbf{D}}. \tag{2}$$

If the preference relation $\prec=$ or its representing functional $\mu$ is known, solving the *direct problem* (2) may be implemented by standard optimization methods. However, preferences of different decision-makers may significantly differ, that is why the functional $\mu$ in (2) should be carefully calibrated prior to solving (2). The calibration might be based on previous decision-making experience for the decision-maker, as well as on some normative principles, which seem relevant for the problem. The calibration constitutes an *inverse problem*, which is considered in brief in what follows.

Let $\mathbf{M}$ be the set of all risk measures $\mu : \mathbf{F} \to \mathbf{R}$. It is clear that if no additional information on $\mu$ is available, then one should determine the value of $\mu$ for all $F \in \mathbf{F}$. Now let $\mathbf{N} \subseteq \mathbf{M}$ be a family of risk measures, and it is known in advance that the risk measure of interest $\mu$ belongs to $\mathbf{N}$. The additional information may allow reducing computational efforts necessary to build the risk measure $\mu$. To make the statement precise we need a concept of characteristic class.

**Definition 1**. A subset of distributions $\mathbf{G} = \mathbf{G}(\mathbf{N}) \subseteq \mathbf{F}$ is called a characteristic class of a family of risk measures $\mathbf{N}$, if for any functional $\nu : \mathbf{G} \to \mathbf{R}$ there exists at most one risk measure $\mu \in \mathbf{N}$ such that $\mu(F) = \nu(F)$ for all $F \in \mathbf{G}$.

In other words, on the one hand, some functionals $\nu : \mathbf{G} \to \mathbf{R}$ are restrictions of risk measures $\mu \in \mathbf{N}$ onto $\mathbf{G}$, and on the other hand, such a continuation $\mu$ of $\nu$ from $\mathbf{G}$ to the whole $\mathbf{F}$ is unique (if exists at all). Thus the additional information $\mu \in \mathbf{N}$ allows reducing the inverse problem from $\mathbf{F}$ to $\mathbf{G}$. This might be a great simplification if a characteristic class $\mathbf{G}$ is small. Note that a characteristic class for a family of risk measures $\mathbf{N}$ in general is not unique.

A similar concept may be introduced for preference relations. Let $\mathbf{P}$ be the set of all preference relations on $\mathbf{F}$, corresponding to risk measures $\mu \in \mathbf{M}$ via (1). Recall that a preference relation $\prec=$ is also a subset $P \subseteq \mathbf{F} \times \mathbf{F}$. Let $\mathbf{Q} \subseteq \mathbf{P}$ be a family of preference relations on $\mathbf{F}$.

**Definition 2**. A subset of distributions $\mathbf{G} = \mathbf{G}(\mathbf{Q}) \subseteq \mathbf{F}$ is called a characteristic class of a family of preference relations $\mathbf{Q}$, if for any preference relation $Q \subseteq \mathbf{G} \times \mathbf{G}$ there exists at most one preference relation $P \in \mathbf{Q}$ such that $Q \subseteq P$.

In other words, any preference relation possesses at most one continuation from a characteristic class $\mathbf{G}$ onto the whole set $\mathbf{F}$ within the preferences family $\mathbf{Q}$.

**Families of risk measures and classes of distributions**

Now let us define some families of risk measures and classes of distributions, which we will use in the sequel. For $F, G \in \mathbf{F}$ we say that $G$ stochastically dominates $F$, if $F(x) \geq G(x)$ for all real $x$; this ordering is denoted by $F \leq_1 G$; we also write $F <_1 G$ if $F \leq_1 G$ and $F \neq G$. A preference relation $\prec=$ is called (strictly) monotone with respect stochastic dominance, if $F <_1 G$ implies $F \prec G$. A risk measure $\mu \in \mathbf{M}$ is called (strictly) monotone (with respect to stochastic dominance), if $F <_1 G$ implies $\mu(F) < \mu(G)$. Note that monotonicity of a preference is equivalent to monotonicity of the representing (via (1)) risk measure $\mu$. Throughout the paper we will assume that all preference relations in $\mathbf{P}$ and all risk measures in $\mathbf{M}$ are *strictly monotone*. In other words, we will restrict our attention to money-loving decision-makers.

Let $U : \mathbf{R} \to \mathbf{R}$ be a real function. A risk measure

$$\rho_U(F) = \int_{-\infty}^{\infty} U(x)dF(x), \qquad F \in \mathbf{F} \tag{3}$$

is called an *expected utility* functional. The corresponding preference relation $\prec=$ satisfies the axioms of von Neumann and Morgenstern [3]. The preference relation and the expected utility functional are strictly monotonic, if the utility function $U$ is increasing. Denote $\mathbf{M}_u$ the family of all monotone expected utility functionals, and $\mathbf{P}_u$ - the corresponding family of all monotone preference relations on $\mathbf{F}$.

Let $g : [0,1] \to [0,1]$ be a distortion function, that is, an increasing function with $g(0) = 0$ and $g(1) = 1$. A risk measure

$$\pi_g(F) = -\int_0^1 F^{-1}(v)dg(1-v), \qquad F \in \mathbf{F} \tag{4}$$

is called a distorted probability measure [4]. This measure is monotonic, as is the corresponding preference relation. Denote $\mathbf{M}_d$ the family of all distorted probability functionals, and $\mathbf{P}_d$ - the corresponding family of all preference relations on $\mathbf{F}$.

Finally, we will need the combined functional [5] defined by

$$\mu_{U,g}(F) = -\int_0^1 U(F^{-1}(v))dg(1-v), \quad F \in \mathbf{F}, \tag{5}$$

where utility function $U$ and distortion function $g$ satisfy the above conditions. The functional and the corresponding preference relation are also monotonic. Denote $\mathbf{M}_c$ the family of all combined functionals, and $\mathbf{P}_c$ - the corresponding family of all preference relations on $\mathbf{F}$. Note that expected utility functional is a special case of (5) obtained with $g(v) \equiv v$, and distorted probability functional is a special case of (5) corresponding to

$U(x) \equiv x$. Thus the combined functional actually combines expected utility with distorted probability.

For $a \in R$, $p \in (0,1)$ denote $W_a$ a degenerate distribution function, which means that the corresponding random variable $X_a$ is completely defined by $P(X_a = a) = 1$, and $B_{a,p}$ - the Bernoulli distribution function, which means that the corresponding random variable $Y_{a,p}$ is completely defined by $P(Y_{a,p} = a) = p$, $P(Y_{a,p} = 0) = 1 - p$. For $A \subseteq \mathbf{R}$ denote also $\mathbf{W}_A = \{W_a, a \in A\}$ a class of degenerate distributions, and $\mathbf{B}_A = \{B_{a,p}, a \in A, p \in (0,1)\}$ - a class of Bernoulli distributions. We will also make use of the Bernoulli distributions $B_p$, $p \in (0,1)$, with corresponding random variable $Y_p$ satisfying $P(Y_p = 1) = p$, $P(Y_p = -1) = 1 - p$. Denote $\mathbf{B} = \{B_p, p \in (0,1)\}$ the set of all such distributions. Values of functionals (3)-(5) on degenerate and Bernoulli distributions are presented below:

$$\rho_U(W_a) = U(a), \qquad \begin{cases} \rho_U(B_{a,p}) = (1-p)U(0) + pU(a) \\ \rho_U(B_p) = (1-p)U(-1) + pU(1) \end{cases} \tag{6}$$

$$\pi_g(W_a) = a, \qquad \begin{cases} \pi_g(B_{a,p}) = ag(p) \\ \pi_g(B_p) = -1 + 2g(p) \end{cases} \tag{7}$$

$$\mu_{U,g}(W_a) = U(a), \qquad \begin{cases} \mu_{U,g}(B_{a,p}) = U(0) + (U(a) - U(0))g(p) \\ \mu_{U,g}(B_p) = U(-1) + (U(1) - U(-1))g(p) \end{cases}, \tag{8}$$

**Characteristic classes for families of preferences**

In [6] characteristic classes for families of expected utilities, distorted probabilities and combined functionals have been calculated. These are $\mathbf{G}(\mathbf{M}_u) = \mathbf{W}_\mathbf{R}$, $\mathbf{G}(\mathbf{M}_d) = \mathbf{B}_{\{1\}}$ and $\mathbf{G}(\mathbf{M}_c) = \mathbf{W}_\mathbf{R} \cup \mathbf{B}_{\{1\}}$. Here we will calculate characteristic classes for families of corresponding preference relations; these occur to significantly differ from their functional counterparties.

**Theorem 1**. *The characteristic class of the von Neumann – Morgenstern family of preference relations is equal to* $\mathbf{G}(\mathbf{P}_u) = \mathbf{W}_\mathbf{R} \cup \mathbf{B}_\mathbf{R} \cup \mathbf{B}$.

**Proof**. Since any preference from $\mathbf{P}_u$ can be represented by an expected utility functional, it suffices to show that defining preference relation on $\mathbf{G}(\mathbf{P}_u)$ completely defines the corresponding utility function. Recall that positive affine transform of a utility function does not change the preference it represents, so the values of $U$ at two points may be chosen at will, say, $U(0) = 0$ and $U(1) = 1$. With this choice (6) gives $\rho_U(W_a) = U(a)$ and $\rho_U(B_{a,p}) = pU(a)$ for $a \in \mathbf{R}$, $p \in (0,1)$, in particular, $\rho_U(W_1) = 1$.

Next, for $a > 1$ consider all Bernoulli distributions $B_{a,p}$ such that $B_{a,p} \sim W_1$. This well defines the function $h : (1, \infty) \to (0,1)$ satisfying $B_{a,h(a)} \sim W_1$. Monotonicity of the preference implies that $h$ decreases,

$$\lim_{a\to 1} h(a) = 1\,, \quad \lim_{a\to\infty} h(a) = 0\,,$$

which gives

$$U(a) = 1/h(a)\,,\ a \in (1,\infty)\,. \tag{9}$$

Next, consider relation $W_a \sim B_{1,p}$ for $a \in (0,1)$. The relation well defines the function $r:(0,1)\to(0,1)$ such that $W_a \sim B_{1,r(a)}$, $a \in (0,1)$, or $U(a) = r(a)U(1) = r(a)$, thus

$$U(a) = r(a)\,,\ a \in (0,1)\,. \tag{10}$$

Expressions (9), (10) together with initial values at $a = 0$ and $a = 1$ completely define the utility function $U$ on $[0,\infty)$.

Now consider calculation of $U$ on the negative half-line. First note that there exists the unique value $p_0 \in (0,1)$ such that $B_{p_0} \sim W_0$. This gives the value

$$U(-1) = -\frac{p_0}{1-p_0}\,. \tag{11}$$

The rest is calculated similarly to the case of the positive half-line. There exist functions $h_-:(-\infty,-1)\to(0,1)$ and $r_-:(-1,0)\to(U(-1),0)$ such that

$$U(a) = U(-1)/h_-(a)\,, \quad a \in (-\infty,-1)\,, \quad U(a) = r_-(a)U(-1)\,, \quad a \in (-1,0)\,. \tag{12}$$

Thus (11) and (12) completely define $U$ on the negative half-line, and the proof is complete.

**Theorem 2**. *The characteristic class of the distorted probability preference relation is equal to* $\mathbf{G}(\mathbf{P}_d) = \mathbf{W}_{(0,1)} \cup \mathbf{B}_{\{1\}}$.

**Proof**. Since the preference is completely defined by a distortion function $g$, it suffices to calculate it on $(0,1)$. For any $p \in (0,1)$ the preference implies the value $h(p) \in (0,1)$ such that $\mathbf{B}_{1,p} \sim \mathbf{W}_{h(p)}$, which together with (7) gives $g(p) = h(p)$, $p \in (0,1)$. The proof is complete.

**Conclusion**

The concept of characteristic class, introduced in the paper, provides a tool for solving inverse problems of risk theory. The tool appears to have rather simple form; the characteristic classes for commonly used families of risk measures and preferences contain at most diatomic distributions. This gives rise to a hope for successful combining of characteristic classes with experiment planning framework, to effectively extract preference information via polls or observations of actual decision-making process.

Characteristic classes of preference relations occur wider than characteristic classes of related risk measures. This is natural, since values of risk measures are usually unobservable, while preference relation may be easily observed in experiments.

The concept of characteristic class deserves developing in a number of directions. It would be quite useful to find a method for constructing a minimal, perhaps even countable, characteristic class. Another interesting topic is building lattice approximate representations of risk measures using characteristic classes.

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