MODELING COHERENT PREFERENCE RELATIONS IN DECISION-MAKING UNDER RISK

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ABSTRACT
Coherent risk measures proved to be a useful tools in financial risk management and decision-making under risk. Their limitations are relaxed by using generalized coherent risk measures. The present paper is devoted to establishing a representation theorem for generalized coherent risk measures, which gives rise to algorithms of calculation of their values.

KEY WORDS
Decision, control, risk, representation, coherent

1 Introduction

In decision-making under uncertainty and risk people usually cannot select a single criterion for choosing a decision, so the problem is normally set and solved in a multi-criteria framework. This feature makes it difficult to quantify the problem at hand and to use elements of automatic control. There is a great need in tools for quantifying human preferences over risky outcomes in a form of real-valued functionals. The latter allows reducing decision-making problem to standard optimization problems.The very quantification process is a matter of representation theory.

The first representation result of the sort was presented in the seminal book [1]. It was shown there that any linear preference relation over risky outcomes (lotteries) is represented by an expected utility functional. A general representation result for nonlinear preference relations was established in [2]. Later on the search for representation tools has led to introducing classes of functionals such as distorted probability functionals [3], combined functionals [4], and more general coherent risk measures [5].

Further research was directed to generalization of coherent risk measures. In [6] the homogeneity and subadditivity requirements were relaxed and substituted by the convexity one. In [7] an attempt of generalization in a somewhat different direction was made; the attempt was finished in [8, 9].

The present paper is devoted to studying properties of the generalized coherent risk measures introduced in [9]. The paper is organized as follows. Section 2 introduces required concepts and definitions. Section 3 contains main results of the paper. The conclusion section finalized the discussion.

2 Risk and risk measures

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, and $\mathcal{X}$ be the set of all random variables, that is, Borel measurable mappings from Ω to $\mathbf{R}$. In the present paper we will use the term ”risk” for elements of $\mathcal{X}$, and the term ”risk measure” for real-valued functionals defined on $\mathcal{X}$. For the sake of clarity of presentation, we will restrict consideration here by the case of finite sample space: $|\Omega| = n < \infty$. The general case requires more complicated technical tools, yet all the stated results remain valid.

Due to the finiteness of Ω the space $\mathcal{X}$ essentially coincides with $\mathbf{R}^n$, so any risk $X \in \mathcal{X}$ may be represented by an n-tuple $X = (X_1, \ldots, X_n)$. Introduce the usual component-wise order $X \leq Y$ if $X_i \leq Y_i$, $i = 1, \ldots, n$. In [5] the following properties of risk measures f were considered.

- f is called monotone if $X \leq Y$ implies $f(x) \leq f(Y)$;
- f is called positive homogeneous if $f(\lambda X) = \lambda f(X)$ for $\lambda \geq 0$;
- f is call super-additive if $f(X + Y) \geq f(X) + f(Y)$;
- f is called translation invariant if $f(X + aI_\Omega) = f(X) + a$, $a \in \mathbf{R}$.

A functional f is called coherent risk measure if it possesses all the four properties just listed (to be precise, in [5] a functional $(-f)$ was called coherent, but that does not matter much, because switching to and from requires just simple properties reformulation).

Denote C_+ and C_- the non-negative, non-positive, and negative cones of $\mathcal{X}$:

$$C_+ = \{X \in \mathcal{X} : X \geq 0\}, \quad C_- = \{X \in \mathcal{X} : X \leq 0\},$$

$$C_{--} = \{X \in \mathcal{X} : X < 0\}.$$

Next define the set of admissible risks in a manner similar to that of [5]. We will call $A \in \mathcal{X}$ a set of admissible risks if it is a closed convex cone satisfying the following conditions:

$$A \supseteq C_+, \quad A \cap C_{--} = \emptyset. \tag{1}$$

The title is justified by the fact that for a coherent risk measure f the set $\{X : f(X) \geq 0\}$ appears to be the closed convex cone possessing the properties (1).

3 Generalized coherent risk measure

Fix some norm $\|\cdot\|$ in $\mathcal{X}$, denote ∂A the boundary of the cone A, and define a generalized coherent risk measure $f = f_A$ as a functional $f : \mathcal{X} \to \mathbf{R}$ of the form

$$f(X) = (2I_A(X) - 1) \inf_{Y \in \partial A} \|X - Y\|, \tag{2}$$

where I_A stands for indicator function of a set A; $I_A(X) = 1$ for $X \in A$ and $I_A(X) = 0$ elsewhere.

Denote $\mathcal{X}^*$ the set of linear bounded functionals on $\mathcal{X}$ (the dual space),

$$\|g\|_* = \sup_{\|X\|=1} g(X)$$

the norm in this space, and consider the cone A^*, which is dual to the cone A:

$$A^* = \{g \in \mathcal{X} : \ g(X) \geq 0, \ X \in A\}.$$

Denote S^* the unit sphere in $\mathcal{X}^*$:

$$S^* = \{g \in \mathcal{X}^* : \ \|g\|_* = 1\},$$

and

$$A_1^* = A^* \cap S^*$$

their intersection. Now we're ready to state the representation theorem.

Theorem 1 Let f be a generalized coherent risk measure defined by a set of admissible risks A and a norm $\|\cdot\|$. Then the following representation is valid:

$$f_A(X) = \inf_{g \in A_1^*} g(X), \ X \in \mathcal{X}. \tag{3}$$

Inverse is also true: if $A_1^* \subseteq C_+^*$ is a set of nonnegative functionals possessing unit norm, then (3) defines a generalized coherent risk measure.

The proof of the theorem 1 is based on the following lemmas. For a linear functional $g \in \mathcal{X}^*$ denote $L_g^+ = \{X \in \mathcal{X} : g(x) \geq 0\}$, $L_g^- = \{X \in \mathcal{X} : g(x) \leq 0\}$ the half-spaces generated by that functional, and $L_g^0 = \{X \in \mathcal{X} : g(x) = 0\}$ the corresponding hyperplane.

Lemma 1

$$A = \bigcap_{g \in A^*} L_g^+ = \bigcap_{g \in A_1^*} L_g^+. \tag{4}$$

Proof The first equality follows from the fact that $\mathbf{R}^n$ is reflexive, so the second dual cone A^{**} coincides with the original cone A. The second equality is the consequence of the fact that for $a > 0$ half-spaces L_g^+ and L_{ag}^+ coincide, so it is sufficient to replace each ray of the cone A^* with one of its point, say $g/\|g\|$ possessing unit norm.

The following lemma is dual for lemma 1, its proof is quite similar.

Lemma 2

$$\overline{A^c} = \bigcup_{g \in A^*} L_g^- = \bigcup_{g \in A_1^*} L_g^-. \tag{5}$$

The next lemma characterizes the distance from a point X to a hyper-plane in terms of the functional defining that hyper-plane.

Lemma 3 For $X \in \mathcal{X}$, $g \in \mathcal{X}^*$, $\|g\| = 1$ we have

$$d(X, L_g^0) = |g(X)|. \tag{6}$$

Proof It is sufficient to consider the case $g(X) > 0$. The case $g(X) < 0$ is quite symmmetric, and for $g(X) = 0$ the equality clearly holds. Since $\|g\| = 1$, we have $g(Z) \leq \|Z\|$, $Z \in \mathcal{X}$. In particular, for $Y \in L_g^0$ we have $X = (X - Y) + Y$, so $g(X) = g(X - Y) + g(Y) = g(X - Y)$ and

$$g(X) \leq \|X - Y\|, \ Y \in L_g^0.$$

Taking infimum by Y we have

$$g(X) \leq d(X, L_g^0). \tag{7}$$

Now show that $g(X)$ cannot be strictly less than $d(X, L_g^0)$. By definition we have

$$1 = \|g\|_* = \sup_{Z \neq 0} \frac{|g(Z)|}{\|Z\|} = \sup_{g(Z)>0} \frac{g(Z)}{\|Z\|}.$$

Thus there exists a sequence Z_k such that $g(Z_k) > 0$, $k = 1, 2, \ldots$ and $g(Z_k)/\|Z_k\| \to 1$ as $n \to \infty$. Denote $W_k = g(X)Z_k/g(Z_k)$ and $Y_k = X - W_k$. Clearly $g(Y_k) = 0$ and

$$\|X - Y_k\| = \|W_k\| = g(X) \frac{\|Z_k\|}{g(Z_k)} \to g(X) \text{ as } n \to \infty.$$

Thus the inequality $g(X) < d(X, L_g^0)$ is impossible indeed.

Proof of the Theorem 1 Let $X \in A^c$ first. In this case

$$d(X, \partial A) = d(X, A). \tag{8}$$

By lemma 1 we have

$$d(X, A) = \sup_{g \in A_1^*} d(X, L_g^+).$$

The elements $g \in A_1^*$ for which $g(X) \geq 0$ does not affect the right hand side of the last equality ($X \in L_g^+$ for such g), so we can calculate supremum over $B_-(X) = \{g \in A_1^* : g(X) < 0\}$:

$$d(X, A) = \sup_{g \in B_-(X)} d(X, L_g^+).$$

For $g \in B_-(X)$ we have $d(X, L_g^+) = d(X, L_g^0)$, so

$$d(X, A) = \sup_{g \in B_-(X)} d(X, L_g^0).$$

Using lemma 3 we get

$$
\begin{aligned}
d(X, A) &= \sup_{g \in B_-(X)} |g(X)| \\
&= \sup_{g \in B_-(X)} (-g(X)) \\
&= -\inf_{g \in B_-(X)} g(X).
\end{aligned}
$$

Now using definition (2) we have

$$f(X) = \inf_{g \in B_-(X)} g(X).$$

Since the right hand side is non-positive, expanding the domain from $B_-(X)$ to A_1^* (thus adding the X's with $g(X) \geq 0$) would not change the left hand side, so

$$f(X) = \inf_{g \in A_1^*} g(X),$$

as required. The case $X \in A$ is considered similarly using the lemma 2. Inverse statement of the theorem is established by choosing

$$A = \bigcap_{g \in A_1^*} L_g^+.$$

The proof is complete.

Note that coherent risk measures are indeed special case of the generalized ones, corresponding to the norm $\| \cdot \|_\infty$ in $\mathbf{R}^n$.

Note also that any generalized coherent risk measure is monotone, positive homogeneous and superadditive. As for translation invariance, the property is present only in case of the infinity norm. In other cases translation behavior of the functionals becomes more tricky, and will be studied in further papers.

4 Conclusion

The generalized coherent risk measures studied in the paper provide a more flexible and rich set of tools for building optimization problems in decision-making under risk, than it was accessible with coherent risk measures. The representation theorem proved in the paper gives a direct tool for numerical calculation of any risk measure in the class. On the other hand using these new tools may require advanced computational techniques and more powerful hardware, which limit their usage to some extent. Possible applications of the technique include financial risk calculation and control, and risk management in a broad sense.

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