Measuring Risk Aversion

In Natural Preference Systems

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Outline

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- $(\Omega, \mathcal{A}, \mathbf{P})$: probability space
- $X : \Omega \to \mathbf{R}$: random variable (risk)
- $\mathcal{X}$: set or risks endowed with a norm $\| \cdot \|$
- $I \in \mathcal{X} : \; I(\omega) = 1, \, \omega \in \Omega$
- Order: $X \leq Y$ means $X(\omega) \leq Y(\omega), \; \omega \in \Omega$
- Positive Cone $C_+ = \{X \in \mathcal{X} : \; X \geq 0\}$

- $\|X\|_p = \left(\int_\Omega |X(\omega)|^p \, d\mathbf{P}(\omega)\right)^{1/p}$
- $\|X\|_\infty = \sup_{\omega \in \Omega} |X(\omega)|$
- Energy norms
- Minkowski norms

Outline

- Axioms
  - Completeness: $X \preceq Y$ or $Y \preceq X$ **for each pair** $X, Y \in \mathcal{X}$
  - Transitivity: $X \preceq Y,\ Y \preceq Z \Longrightarrow X \preceq Z$
- Parts
  - Symmetric part $X \sim Y \Longleftrightarrow X \preceq Y,\quad Y \preceq X$
  - Asymmetric part $X \prec Y \Longleftrightarrow X \preceq Y,\ Y \not\preceq X$
- Representation
  - $X \preceq Y \Longleftrightarrow f(X) \leq f(Y)$

- Classes
  - Upper class $X \in \mathcal{X} \implies U(X) = \{Y : X \preceq Y\}$
  - Lower class $X \in \mathcal{X} \implies L(X) = \{Y : Y \preceq X\}$
  - Equivalence class
    $X \in \mathcal{X} \implies K(X) = U(X) \cap L(X) = \{Y : Y \sim X\}$
- Mapping
  - Partitioning $C : \mathcal{X} \to \mathcal{X}/\sim; X \mapsto K(X)$

- $\preceq$ is **Finite** if $\forall Y \in \mathcal{X}\ \exists a < b : aI \in L(Y),\ bI \in U(Y)$
- $\preceq$ is **Monotone** if $X \leq Y \implies X \preceq Y$
- $\preceq$ is **Strictly Monotone** if $X \leq Y,\ X \neq Y \implies X \prec Y$
- $\preceq$ is **Lower Semicontinuous at** $Y \in \mathcal{X}$ if it is monotone and for any countable set $A \subseteq U(Y)$ we have $\inf_{X \in A} X \in U(Y)$
- $\preceq$ is **Upper Semicontinuous at** $Y \in \mathcal{X}$ if it is monotone and for any countable set $A \subseteq L(Y)$ we have $\sup_{X \in A} X \in L(Y)$
- $\preceq$ is **Continuous** if it is Lower and Upper Semicontinuous at each $Y \in \mathcal{X}$
- $\preceq$ is **Unsaturated** if $\forall X, Y \in \mathcal{X}\ \exists c, d \in \mathbf{R} :\ X + cI \prec Y \prec X + dI$

Definition 1
Continuous, finite, strictly monotone, unsaturated preference relation is called **natural**

Theorem 1
Let $\preceq$ be natural. Then for any $X \in \mathcal{X}$ we have

$$U(X) = \bigcup_{Y \in K(X)} (Y + C_+)$$

Theorem 2
Let $\preceq$ be natural. Then there is a representative functional $f : \mathcal{X} \to \mathbf{R}$; in particular it may be chosen to satisfy

$$f(cI) = c, \; c \in \mathbf{R}$$

Outline

- Given an utility function $U: \mathbf{R} \to \mathbf{R}$
- Expected utility functional $u(X) = \mathbf{E}(X),\ X \in \mathcal{X}$
- $A(x) = -\frac{U''(x)}{U'(x)},\ x \in \mathbf{R}$

Outline

- Distortions $\mathcal{X}_0 = \{X \in \mathcal{X} : \mathbf{E}X = 0\}$
- Qualitative Risk Aversion $xI + \Delta \prec xI, \;\; \Delta \in \mathcal{X}_0, \; x \in \mathbf{R}$
- Norm calibration $\|I\| = 1$
- Unit Ball $B = \{X \in \mathcal{X} : \|X\| \leq 1\}$
- Unit Ball Section $B_0 = \mathcal{X}_0 \cap B$
- Risk Aversion Function
  $r(x) = \sup_{\Delta \in B_0}(f(xI) - f(xI + \Delta)), \; x \in \mathbf{R}$

Outline

Theorem 3

Let $\preceq$ be natural, represented by f. Then

$$r(x) = x - \sup\{y : \; U(yI) \supseteq xI + B_0\}, \; x \in \mathbf{R}$$

Theorem 4

Let $\preceq_1, \preceq_2$ be natural, and $U_1(xI) \subseteq U_2(xI), \; x \in \mathbf{R}$. Then $r_1(x) \geq r_2(x), \; x \in \mathbf{R}$.

Energy Norm

- Energy norm $\|X\|_A = \sqrt{X'AX}$
- Simple space $|\Omega| = 2$, $\mathbf{P}(\omega) = 0.5, \ \omega \in \Omega$

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad a = \frac{1+c}{4}, \ b = \frac{1-c}{4}, \ c > 0$$

- $r(x) = 1/\sqrt{c}$

Coherent Risk Measure

$$f(X) = \inf_{Q \in \mathcal{Q}} \mathbf{E}_Q X$$

- Simple space $|\Omega| = 2$, $\mathbf{P} = \left(\frac{5}{8}, \frac{3}{8}\right)$,
- Norm $\|X\| = \max_i |X_i|$
- $\mathcal{Q} = \left\{\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{3}{4}, \frac{1}{4}\right)\right\}$

$$\mathcal{X}_0 : \quad \frac{5}{8} X_1 + \frac{3}{8} X_2 = 0, \quad \partial B_0 = \left\{ \left(\frac{3}{5}, -1\right), \left(-\frac{3}{5}, 1\right) \right\}$$

$$r = \max_{Y \in \partial B_0} \left(|f(Y)|\right) = \max\left(\frac{1}{5}, \frac{7}{15}\right) = \frac{7}{15}$$

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