Risk Aversion Functional

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Outline

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Risks Space

Basic Concepts

- $(\Omega, \mathcal{A}, \mathbf{P})$: probability space
- $X : \Omega \to \mathbf{R}$: random variable (risk)
- $\mathcal{X}$: set or risks endowed with a norm $\|\cdot\|$
- $I \in \mathcal{X} : \; I(\omega) = 1, \, \omega \in \Omega$
- Order: $X \leq Y$ means $X(\omega) \leq Y(\omega), \; \omega \in \Omega$
- Positive Cone $C_+ = \{X \in \mathcal{X} : X \geq 0\}$

Norm examples

- $\|X\|_p = \left(\int_\Omega |X(\omega)|^p \, d\mathbf{P}(\omega)\right)^{1/p}$
- $\|X\|_\infty = \sup_{\omega \in \Omega} |X(\omega)|$
- Energy norms
- Minkowski norms

Risks Space

Example

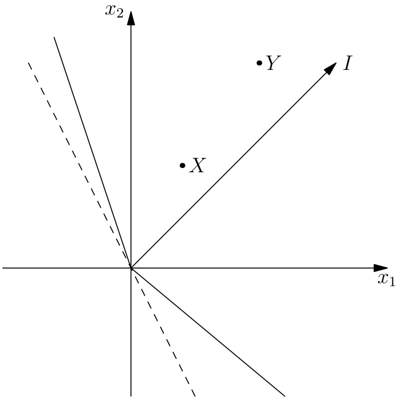

Figure: Basic concepts, $n = 2$

Outline

Preference Relation

Axioms and Representation

- Axioms
  - Completeness: $X \preceq Y$ or $Y \preceq X$ **for each pair** $X, Y \in \mathcal{X}$
  - Transitivity: $X \preceq Y,\ Y \preceq Z \Longrightarrow X \preceq Z$
- Parts
  - Symmetric part $X \sim Y \Longleftrightarrow X \preceq Y, \quad Y \preceq X$
  - Asymmetric part $X \prec Y \Longleftrightarrow X \preceq Y,\ Y \npreceq X$
- Representation
  - $X \preceq Y \Longleftrightarrow f(X) \leq f(Y)$

Preference Relation

Classes

- Classes
  - Upper class $X \in \mathcal{X} \Longrightarrow U(X) = \{Y : X \preceq Y\}$
  - Lower class $X \in \mathcal{X} \Longrightarrow L(X) = \{Y : Y \preceq X\}$
  - Equivalence class $X \in \mathcal{X} \Longrightarrow K(X) = U(X) \cap L(X) = \{Y : Y \sim X\}$
- Mapping
  - Partitioning $C : \mathcal{X} \to \mathcal{X}/\sim; X \mapsto K(X)$

Preference Relation

Properties

- $\preceq$ is **Finite** if $\forall Y \in \mathcal{X}\ \exists a < b : aI \in L(Y),\ bI \in U(Y)$
- $\preceq$ is **Monotone** if $X \leq Y \implies X \preceq Y$
- $\preceq$ is **Strictly Monotone** if $X \leq Y, X \neq Y \implies X \prec Y$
- $\preceq$ is **Lower Semicontinuous at** $Y \in \mathcal{X}$ if it is monotone and for any countable set $A \subseteq U(Y)$ we have $\inf_{X \in A} X \in U(Y)$
- $\preceq$ is **Upper Semicontinuous at** $Y \in \mathcal{X}$ if it is monotone and for any countable set $A \subseteq L(Y)$ we have $\sup_{X \in A} X \in L(Y)$
- $\preceq$ is **Continuous** if it is Lower and Upper Semicontinuous at each $Y \in \mathcal{X}$
- $\preceq$ is **Unsaturated** if $\forall X, Y \in \mathcal{X}\ \exists c, d \in \mathbf{R} :\ X + cI \prec Y \prec X + dI$

Preference Relation

Natural Preference

Definition 1
Continuous, finite, strictly monotone, unsaturated preference relation is called **natural**

Theorem 1
Let $\preceq$ be natural. Then for any $X \in \mathcal{X}$ we have

$$U(X) = \bigcup_{Y \in K(X)} (Y + C_+)$$

Theorem 2
Let $\preceq$ be natural. Then there is a representative functional $f : \mathcal{X} \to \mathbf{R}$; in particular it may be chosen to satisfy

$$f(cI) = c, \ c \in \mathbf{R}$$

Outline

Risk Aversion

Concept

- Distortions $\mathcal{X}_0 = \{X \in \mathcal{X} : \ \mathbf{E}X = 0\}$
- Qualitative Risk Aversion $xI + \Delta \prec xI, \quad \Delta \in \mathcal{X}_0, \ x \in \mathbf{R}$
- Norm calibration $\|I\| = 1$
- Unit Sphere $B = \{X \in \mathcal{X} : \|X\| = 1\}$
- Unit Sphere Section $B_0 = \mathcal{X}_0 \cap B$
- Risk Aversion Functional
  $s(x, \Delta) = f(xI) - f(xI + \Delta), \ x \in \mathbf{R}, \ \Delta \in B_0$
- Risk Aversion Functions $R(x) = \sup_{\Delta \in B_0} s(x, \Delta), \ x \in \mathbf{R}$
  $r(x) = \inf_{\Delta \in B_0} s(x, \Delta), \ x \in \mathbf{R}$

Risk Aversion

Illustration

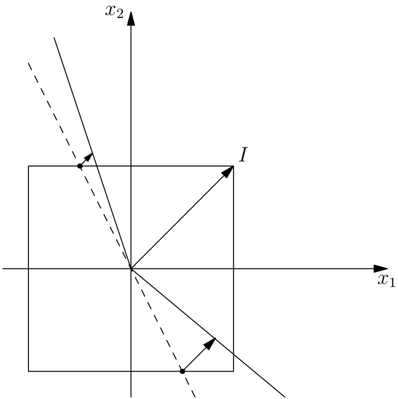

Figure: Risk aversion, $n = 2$

Outline

Risk Aversion

Properties

Theorem 3
Let $\preceq$ be natural, represented by f. Then

$$R(x) = x - \sup\{y : U(yI) \supseteq xI + B_0\}, \; x \in \mathbf{R}$$

Theorem 4
Let $\preceq_1, \preceq_2$ be natural, and $U_1(xI) \subseteq U_2(xI), \; x \in \mathbf{R}$. Then $R_1(x) \geq R_2(x), \; x \in \mathbf{R}$.

Outline

Coherent Risk Measure

$$f(X) = \inf_{Q \in \mathcal{Q}} \mathbf{E}_Q X$$

- Simple space $|\Omega| = 2$, $\mathbf{P} = \left(\frac{2}{3}, \frac{1}{3}\right)$,
- Norm $\|X\| = \max_i |X_i|$
- $\mathcal{Q} = \left\{ \left(\frac{3}{4}, \frac{1}{4}\right), \left(\frac{5}{11}, \frac{6}{11}\right) \right\}$

$$\mathcal{X}_0 : \quad \frac{2}{3}x_1 + \frac{1}{3}x_2 = 0, \quad B_0 = \left\{ \left(\frac{1}{2}, -1\right), \left(-\frac{1}{2}, 1\right) \right\}$$

$$R = \max_{Y \in B_0}(|f(Y)|) = \max(0.13, 0.05) = 0.13$$

Outline

Dimension 3

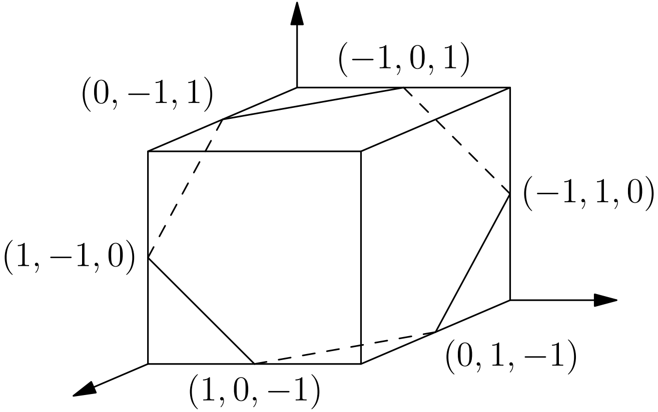

Figure: Risk aversion, $n = 3$

Outline

Dimension ∞

$$\Delta_\varepsilon(\omega) = \begin{cases} 1, & \omega \in \Omega_\varepsilon \\ -\frac{\varepsilon}{1-\varepsilon}, & \omega \in \Omega_\varepsilon^c \end{cases}$$

$$\mathbf{P}(\Omega_\varepsilon) = \varepsilon, \implies \Delta_\varepsilon \in B_0,$$

$$\Delta_\varepsilon + \frac{\varepsilon}{1-\varepsilon} I \in C_+ \implies r(0) \le \frac{\varepsilon}{1-\varepsilon} \implies r(0) = 0$$

Dimension ∞, picture

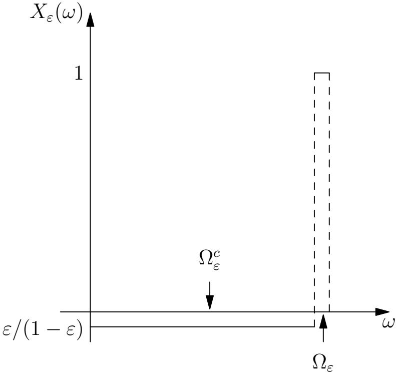

Figure: Risk aversion, lower bound, $n = \infty$

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